Questions 1 to 5 are on Metric and Topological Spaces.
Questions 6 to 9 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 6) – 6 marks

Prove that the subspace

$$H = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 \leq 1,\ 0 < x_2 \leq 1\}$$

of $\mathbf{R}^2$ with the usual topology is not compact, by writing down an appropriate open cover of H and showing that it does not have a finite subcover. [6]

Question 2 (Unit 6) – 10 marks

Let A be the set of non-negative integers, and let $\mathscr{T}$ be the topology on A which consists of all subsets V of A such that either $0 \notin V$ or there exists some positive integer m such that $n \in V$ for all integers $n > m$.

[You need *not* verify that $\mathscr{T}$ is a topology on A.]

(i) Show that if $V \in \mathscr{T}$ and $0 \in V$, then $A - V$ is finite. [4]

(ii) Deduce that $\{A, \mathscr{T}\}$ is a compact space. [6]

Question 3 (Unit 6) – 9 marks

For each of the following subspaces of $\mathbf{R}^2$ with the usual topology, state whether or not it is compact, briefly justifying your answer.

(i) $\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \geq 1\}$

(ii) $\{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1^2 + x_2^2 \leq 1\}$

(iii) $\{(x_1, x_2) \in \mathbf{R}^2 : \frac{1}{2} \leq x_1^2 + x_2^2 \leq 1\}$ [9]

Question 4 (Unit 7) – 12 marks

Let $A = \{a, b, c, d, e\}$ and let $\mathscr{T}$ be the coarsest topology on A which contains the subsets $\{d\}$, $\{b, d, e\}$ and $\{a, b, c, e\}$.

(i) Show that the topological space $\{A, \mathscr{T}\}$ is not connected. [5]

(ii) Show that $\{a, b, c\}$ with the induced topology as a subspace of $\{A, \mathscr{T}\}$ is connected. [5]

(iii) Find the connected components of $\{A, \mathscr{T}\}$. [2]

Question 5 (Unit 7) – 13 marks

For each positive integer n, let K_n be the line segment in $\mathbf{R}^2$ joining $(0, 0)$ to $(1, \frac{1}{n})$, and let L_n be the line segment in $\mathbf{R}^2$ joining $(\frac{1}{n}, 0)$ to $(\frac{1}{n}, 1)$. Put

$$A = \bigcup_{n=1}^{\infty} K_n, \quad B = \{(0, 1)\} \cup K_1 \cup \bigcup_{n=1}^{\infty} L_n.$$

(i) Sketch the subsets A and B of $\mathbf{R}^2$. [2]

(ii) Prove that the subspaces A and B of $\mathbf{R}^2$ with the usual topology are connected. [7]

(iii) Explain briefly why the subspaces A and B of $\mathbf{R}^2$ with the usual topology are not homomorphic. [4]