③

ii) Put $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \{(x_1,x_2) \in \mathbb{R}^2 : |x_2| \le \frac{n}{n+1} x_1\}$

$B = \{(0,0)\} \cup \{(x_1,x_2) \in \mathbb{R}^2 : |x_2| < x_1\}$

We prove that if $(x_1,x_2) \notin B$, then $(x_1,x_2) \notin A$

If $(x_1,x_2) \notin B$, then either

1) $x_1 < 0$

2) $x_1 = x_2$ (See explanation for behaviour at $(0,0)$)

3) $|x_2| > x_1$

Easier to make the subdivision $\{x_1<0$, $|x_2| \ge x_1$

1) $x_1 < 0 \Rightarrow 0 \le |x_2| \le \frac{n}{n+1} x_1 < 0$; a contradiction

$\therefore (x_1,x_2) \notin A$ for $x_1 < 0$

2) $x_1 = x_2$ ($x_1 = x_2 = 0$ is a soln to $|x_2| \le {}^{n}/_{(n+1)} x_1$ $\therefore (0,0) \in A$ (and B by defn))

If $x_1 = x_2 \ne 0$, then from defn of A_n,

$|x_2| \le \frac{n}{n+1} x_1 = \left(1 - \frac{1}{n+1}\right) x_1 < x_1 = |x_2|$

$\therefore |x_2| < x_1 = |x_2|$ so $x_1 = |x_2| \Rightarrow (x_1,x_2) \notin A_n$.

3) $|x_2| > x_1$ from defn of A_n, $|x_2| {}^{(n+1)}/_{n} \le x_1$

$|x_2| > x_1 \Rightarrow |x_2| > x_2 \frac{(n+1)}{n} = \left(1 + \frac{1}{n}\right) |x_2|$

Divide by $|x_2|$; $\frac{|x_2|}{|x_2|} = 1 > \frac{n+1}{n} = 1 + \frac{1}{n} > 1$

$1 > 1 + \frac{1}{n} > 1$, a contradiction $\therefore |x_2| > x_1 \Rightarrow (x_1,x_2) \notin A$

Hence $(x_1,x_2) \notin A$ if $(x_1,x_2) \notin B$

We proved above that $(0,0) \in A$, and $(0,0) \in B$ by defn. In part ii) we proved that if $|x_2| < x_1$ then there exists $m \in \mathbb{N}$ such that $(x_1,x_2) \in A_m$, and by defn $|x_2| < x_1 \Rightarrow (x_1,x_2) \in B$

so $\{(x_1,x_2) \in \mathbb{R}^2 : |x_2| < x_1\} = \bigcup_{n=1}^{\infty} A_n = A$

$\therefore B = A$ as required.

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