④

2) i) a) $f(A \cap B) \neq f(A) \cap f(B)$

LHS first

$f([-2,-1] \cap [1,2]) = f(\emptyset) = \emptyset$

RHS

$f([-2,-1]) \cap f([1,2]) = \{x \in \mathbb{R} : 1 \leq x \leq 2\} \cap \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

$= \{x \in \mathbb{R} : 1 \leq x \leq 2\} \neq \emptyset = f([-2,-1] \cap [1,2])$.

$\therefore f(A \cap B) \neq f(A) \cap f(B)$

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b) $f(A) = f([-2,-1]) = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.

$f^{-1}[1,2] = \{x \in \mathbb{R} : -2 \leq x \leq -1\} \cup \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

$= \{x \in \mathbb{R} : -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$

$f(B) = f([1,2]) = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

$f^{-1}[1,2] = \{x \in \mathbb{R} : -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$.

(Hence $f^{-1}(f(A)) = f^{-1}(f(B))$).

$= A \cup B$

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ii) Show that if $C, D \subset X$ and $g^{-1}(g(C)) = C$ then $g(C \cap D) = g(C) \cap g(D)$.

If $x \in C \cap D$ then $x \in C$ and $x \in D$

so $g(x) \in g(C)$ and $g(x) \in g(D)$

$\therefore g(x) \in g(C) \cap g(D)$ $\therefore$

$g(C \cap D) \subset g(C) \cap g(D)$

If $g(x) \in g(C) \cap g(D)$ then $g(x) \in g(C)$ and $g(x) \in g(D)$. But $g^{-1}(g(C)) = C$ so if $g(x) \in g(C)$, $x \in g^{-1}(g(C))$, and if $g(x) \in g(D)$ since $g^{-1}(g(D)) = D$, $x \in g^{-1}(g(D))$ — not given

$\therefore x \in g^{-1}(g(C)) \cap g^{-1}(g(D)) = C \cap D$

$\therefore g(x) \in g(C \cap D)$

$g(C) \cap g(D) \subset g(C \cap D)$

If $y \in g(C) \cap g(D)$
then there exists $x \in D$ s.t. $g(x) = y$
As $g(x) \in g(C)$, $x \in g^{-1}(g(C)) = C$
so $x \in C \cap D$ & $y \in g(C \cap D)$
so $g(C) \cap g(D) \subset g(C \cap D)$

Hence $g(C \cap D) \subset g(C) \cap g(D)$

and $g(C) \cap g(D) \subset g(C \cap D)$.

so $g(C \cap D) = g(C) \cap g(D)$.

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