Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 8 are on Geometric Topology.

Metric and Topological Spaces

Question 1 – 13 marks

For each positive integer n, let

$$A_n = \left\{ (x_1, x_2) \in \mathbf{R}^2 : |x_2| \le \frac{n}{n+1} x_1 \right\}.$$

(i) Sketch the subsets A_1, A_2 of $\mathbf{R}^2$. [2]

(ii) Prove that if $(x_1, x_2) \in \mathbf{R}^2$ and $|x_2| < x_1$, then there exists a positive integer m such that $(x_1, x_2) \in A_m$. [5]

(iii) Prove that

$$\bigcup_{n=1}^{\infty} A_n = \{(0,0)\} \cup \{(x_1, x_2) \in \mathbf{R}^2 : |x_2| < x_1\}.$$ [6]

Question 2 – 12 marks

(i) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = |x|, \quad x \in \mathbf{R},$$

and let

$$A = [-2, -1], \quad B = [1, 2].$$

(a) Show that

$$f(A \cap B) \ne f(A) \cap f(B).$$ [3]

(b) Write down the sets $f^{-1}(f(A))$ and $f^{-1}(f(B))$. [3]

$y < \frac{x}{2}$

(ii) Let X and Y be sets, let $g : X \longrightarrow Y$ be a function, and let C and D be subsets of X. Prove that if $g^{-1}(g(C)) = C$, then

$$g(C \cap D) = g(C) \cap g(D).$$ [6]

Question 3 – 12 marks

(i) Let $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$ be given by

$$f(x_1, x_2) = |x_1| + |x_2|, \quad (x_1, x_2) \in \mathbf{R}^2.$$

(a) Write down $f([0,1] \times [1,2])$.

(b) Write down $f^{-1}(\{-1\})$.

(c) Sketch the subset $f^{-1}([1,3])$ of $\mathbf{R}^2$, and draw the subset $[0,1] \times [1,2]$ on your sketch. [6]

(ii) Let $g : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ be given by

$$g(x_1, x_2) = (x_1 x_2,\ |x_1| + |x_2|), \ (x_1, x_2) \in \mathbf{R}^2.$$

Sketch the subset $g^{-1}([0,1] \times [1,3])$ of $\mathbf{R}^2$. [6]