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To find whether (∞,∞) is on the locus, Put $(z,w)=\left(\frac{1}{\tilde{z}},\frac{1}{\tilde{w}}\right)$

$$\frac{1}{\tilde{w}^2}-\frac{2}{\tilde{w}\tilde{z}}+\frac{1}{\tilde{z}^3}+\frac{1}{\tilde{z}^2}-1=0$$

$$\tilde{z}^3-2\tilde{z}^2\tilde{w}+\tilde{w}^2+\tilde{w}^2\tilde{z}-\tilde{w}^2\tilde{z}^3=0$$

Put $\tilde{z}=\tilde{w}=0$ to give $0=0$ $\therefore$ (∞,∞) lies on the locus. $\tilde{z}=0 \Rightarrow \tilde{w}^2=0$ $\therefore$

$\therefore (\infty,\infty)$ has multiplicity 2.

3/3 ✓

ii) Solve for w

$$w=\frac{2z\pm\sqrt{(-2z)^2-4(z^3+z^2-1)}}{2}$$

$$=\frac{2z\pm\sqrt{4z^2-4z^3-4z^2+4}}{2}$$

$$=\frac{2z\pm\sqrt{-4z^3+4}}{2}$$

$$=z\pm\sqrt{1-z^3}$$

$$=z\pm\sqrt{(1-z)(1+z+z^2)}$$ $\therefore z=1$ is one soln

We must solve $z^2+z+1=0$ to find the others

$$z=\frac{-1\pm\sqrt{1-4\cdot1\cdot1}}{2}$$

$$=\frac{-1\pm i\sqrt{3}}{2}$$

$\therefore$ The three values of z which give a unique soln for w are $z=1$, $z=\frac{-1+i\sqrt{3}}{2}$, $z=\frac{-1-i\sqrt{3}}{2}$

the corresponding values for w are $w=1$, $w=\frac{-1+i\sqrt{3}}{2}$, $w=\frac{-1-i\sqrt{3}}{2}$

ie $(z,w)=(1,1)$, $\left(\frac{-1+i\sqrt{3}}{2},\frac{-1+i\sqrt{3}}{2}\right)$

3/3 ✓ $\left(\frac{-1-i\sqrt{3}}{2},\frac{-1-i\sqrt{3}}{2}\right)$