Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 8 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 4) – 14 marks

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

(i) Show that

$$\mathscr{T} = \{\varnothing, \{1\}, \{1,2\}, \{1,3\}, A\}$$

is a topology on A. ∅, b, (bc), c, (ab), B [2]

(ii) Explain why

$$\mathscr{B} = \{\varnothing, \{b\}, \{c\}, \{a,b\}, B\}$$

is not a topology on B. [2]

(iii) Find the coarsest topology $\mathscr{V}$ on B such that $\mathscr{B} \subset \mathscr{V}$. [2]

(iv) Show that $f : A \longrightarrow B$ given by

$$f(1) = f(2) = b, \quad f(3) = a$$

is $(\mathscr{T}, \mathscr{V})$-continuous. [3]

(v) Show that $g : A \longrightarrow B$ given by

$$g(1) = a, \quad g(2) = g(3) = b$$

is not $(\mathscr{T}, \mathscr{V})$-continuous. [2]

(vi) State whether or not $h : A \longrightarrow B$ given by

$$h(1) = c, \quad h(2) = h(3) = a$$

is $(\mathscr{T}, \mathscr{V})$-continuous, and justify your answer. [3]

Question 2 (Unit 4) – 11 marks

Let $\mathbf{Z}$ and $\mathbf{E}$ denote the sets of all integers, and all even integers, respectively. For each positive integer n, let

$$V_n = \mathbf{E} \cup \{k \in \mathbf{Z} : |k| \geq n\}.$$

(i) Prove that the collection $\mathscr{V}$ of subsets of $\mathbf{Z}$ consisting of $\varnothing$ and the sets V_n, $n = 1, 2, \ldots$, is a topology on $\mathbf{Z}$. [5]

(ii) Let $H = \{-4, -2, 3, 5\}$.

(a) Show that $\{-4, -2\}$ and $\{-4, -2, 5\}$ are open sets of the topology induced by $\mathscr{V}$ on H. [3]

(b) Determine the topology induced by $\mathscr{V}$ on H. [3]

Question 3 (Unit 5) – 10 marks

Let $\mathscr{V}$ be the topology on the set $\mathbf{Z}$ of all integers introduced in Question 2 above.

(i) Prove that $\{-1, 1\}$ is closed in $\{\mathbf{Z}, \mathscr{V}\}$. [2]

(ii) Find all closed sets in $\{\mathbf{Z}, \mathscr{V}\}$. [4]

(iii) Find the closure of $\{-1, 0, 1\}$ in $\{\mathbf{Z}, \mathscr{V}\}$. [2]

(iv) State whether or not $\{\mathbf{Z}, \mathscr{V}\}$ is a Hausdorff space, and justify your answer. [2]