Question 4 (Unit 3) – 19 marks

Let $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined as follows:

if $x \in \mathbf{R}$, $y \in \mathbf{R}$, then

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 2|x| + 2|y| + |x-y|, & \text{if } x \neq y. \end{cases}$$

(i) Prove that d is a metric on $\mathbf{R}$. *[8]*

(ii) In the metric space $M = \{\mathbf{R}, d\}$, determine the open balls:

(a) $B_1(0)$;

(b) $B_1(1)$. *[4]*

(iii) Show that if $a \in \mathbf{R}$ and $a \neq 0$, then there exists a positive real number r such that, in the metric space M, $B_r(a) = \{a\}$. *[2]*

(iv) For each of the following subsets of $\mathbf{R}$, state whether or not it is open in M, justifying your answer:

(a) $[-\frac{1}{3}, 1]$;

(b) $[0, 1]$. $\frac{N_1}{3} = \frac{N_2}{2}$ *[5]*

Geometric Topology

Question 5 (Unit 2) – 13 marks

This question concerns closed surfaces with a regular subdivision consisting of topological k-gons meeting j at a vertex. Let V be the number of vertices, E the number of edges, and F the number of faces in such a regular subdivision, and let χ be the Euler characteristic of the surface.

(i) Write down a relationship connecting k, F and E. *[1]*

(ii) Write down a relationship connecting j, V and E. *[1]*

(iii) Hence write down an equation connecting j, k, F and χ. *[2]*

For the rest of this question, assume that $F = 1$, and that $j \geq 2$.

(iv) Express χ as a function of k and j. *[1]*

(v) Show that there are no possible values of j, k, V and E when $\chi = 2$. *[2]*

(vi) Show that when $\chi = 1$, there are infinitely many solutions to your equation in part (iv), and write the corresponding values of j, k, V and E explicitly. *[2]*

(vii) Show that when $\chi = 0$, there are three solutions for k and j to your equation in part (iv) (only two of which correspond to surfaces), and write down the corresponding three sets of values of j, k, V and E explicitly. Explain why one of your answers cannot correspond to a surface. *[4]*

$B_1(\frac{1}{2})$ $\quad \cancel{2}\, 1 + 2|y| + |\frac{1}{2} + y| < 2$

$2y + y - \frac{1}{2} < 1$

$y < \frac{1}{2}$

$1 + 2y + \frac{1}{2} - y < 2\frac{1}{2}$

$1 +$