Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 8 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 2) – 12 marks

(i) Let A be a non-empty subset of $\mathbf{R}$ which is bounded above, and put $B = \{-x : x \in A\}$. Prove that B is bounded below, and that

$$\inf B = -\sup A. \qquad [5]$$

(ii) Let X be a non-empty subset of $\mathbf{R}$ which is bounded above and below, and put $Y = \{|x| : x \in X\}$. Prove that Y is bounded below, and that

$$0 \leq \inf Y \leq \operatorname{Min}\{|\inf X|, |\sup X|\}. \qquad [7]$$

Question 2 (Unit 2) – 13 marks

(i) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = x^2 + 2x + 3, \quad x \in \mathbf{R}.$$

Show that given $\varepsilon > 0$,

$$|f(x) - f(1)| < \varepsilon$$

whenever $|x - 1| < \delta$, if we take $\delta = \operatorname{Min}\left\{1, \frac{\varepsilon}{4}\right\}$. [5]

(ii) For $g : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$g(x) = x^3 + 2x^2 + 3x + 4, \quad x \in \mathbf{R},$$

given $\varepsilon > 0$, find a $\delta > 0$ such that

$$|g(x) - g(1)| < \varepsilon$$

whenever $|x - 1| < \delta$. [6]

(iii) Deduce that the function $h : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$h(x) = \frac{x^3 + 2x^2 + 3x + 4}{x^2 + 2x + 3}, \quad x \in \mathbf{R},$$

is continuous at 1. [2]

(You may use any result from the unit, provided that a reference is given.)

Question 3 (Unit 3) – 6 marks

For each of the following functions $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$, explain why d is not a metric on $\mathbf{R}$:

(i) $d(x, y) = |x^2 - y^2|$;

(ii) $d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 3|x| + 2|y|, & \text{if } x \neq y; \end{cases}$

(iii) $d(x, y) = |x - y|(|x - y| + 2)$. [6]