**Question 4** (Unit 3) – 20 marks

Let $d : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ be defined as follows: if $x = (x_1, x_2) \in \mathbf{R}^2$ and $y = (y_1, y_2) \in \mathbf{R}^2$ and $x \neq y$, then

$$d(x, y) = \mathrm{Max}\{|x_1|, |x_2|\} + \mathrm{Max}\{|y_1|, |y_2|\},$$

whilst $d(x, y) = 0$ if $x = y$.

(i) Prove that $d$ is a metric on $\mathbf{R}^2$. [8]

(ii) In the metric space $M = \{\mathbf{R}^2, d\}$, determine the open balls:

(a) $B_1((0, 0))$;

(b) $B_1((1, 1))$. [6]

(iii) Prove that the set

$$\{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$$

is open in the metric space $M$. [6]

## Geometric Topology

**Question 5** (Unit 2) – 13 marks

This question concerns surfaces with Euler characteristic $\chi = 2$, 1 or 0. Each surface has a regular subdivision consisting of $k$-gons meeting $j$ at a vertex. Let $V$ be the number of vertices, $E$ the number of edges, and $F$ the number of faces, in such a regular subdivision.

(i) Write down a relationship connecting $k$, $F$ and $E$. [1]

(ii) Write down a relationship connecting $j$, $V$ and $E$. [1]

(iii) Hence write down an equation connecting $j$, $k$, $F$ and $\chi$. [1]

For the rest of the question, assume that $j = k$ and $k \geq 3$.

(iv) Express $k$ as a function of $\chi$ and $F$. [2]

(v) Hence determine possible values of $V$, $E$, $F$ and $k$ when

(a) $\chi = 2$,

(b) $\chi = 1$,

(c) $\chi = 0$. [5]

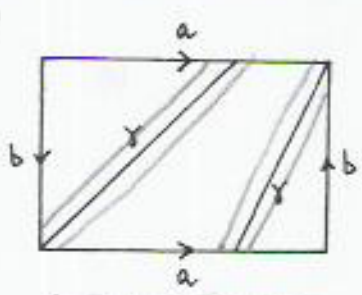
(vi) For each of your answers to (a) and (c) above, give a sketch of a surface or a topological polygon displaying the regular subdivision. [3]

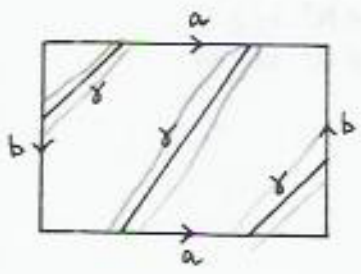
**Question 6** (Unit 2) – 12 marks

*In order to answer part (i) of this question, you will find it helpful to refer to Section 2.4 of the first Exercise Booklet for Geometric Topology.*

(i) Find a neighbourhood for each of the following curves $\gamma$, and mark the neighbourhood on a sketch. In each case state, giving reasons, whether $\gamma$ is orientation-preserving or orientation-reversing.

(a)




reversing

(b)