Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 8 are on Geometric Topology.

Metric and Topological Spaces

Question 1 (Unit 2) – 12 marks

(i) Let A be a non-empty set of real numbers which is bounded above by α. Prove that $\sup A = \alpha$ if and only if for each $\varepsilon > 0$, we can find $a \in A$ such that

$$\alpha - \varepsilon < a \leq \alpha.$$ [4]

(ii) Let A, B be non-empty sets of real numbers which are bounded above, and let C be the set of all numbers of the form $a + b$ where $a \in A$, $b \in B$.

(a) Show that C is bounded above. [2]

(b) Prove that if $\sup A = \alpha$ and $\sup B = \beta$, then $\sup C = \alpha + \beta$. [6]

Question 2 (Unit 2) – 13 marks

(i) Let $f : \mathbf{R} \to \mathbf{R}$ be given by

$$f(x) = 1 + 3x^2, \quad x \in \mathbf{R}.$$

Show that given $\varepsilon > 0$,

$$|f(x) - f(1)| < \varepsilon$$

whenever $|x - 1| < \delta$, if we take $\delta = \mathrm{Min}\left\{1, \frac{\varepsilon}{9}\right\}$. [5]

(ii) For $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = 3x^2 + x + 5, \quad x \in \mathbf{R},$$

given $\varepsilon > 0$, find a $\delta > 0$ such that

$$|g(x) - g(1)| < \varepsilon$$

whenever $|x - 1| < \delta$. [6]

(iii) Deduce that the function $h : \mathbf{R} \to \mathbf{R}$ given by

$$h(x) = \frac{1 + 3x^2}{3x^2 + x + 5}, \quad x \in \mathbf{R},$$

is continuous at 1. [2]

(You may use any result from the unit, provided that a reference is given.)

Question 3 (Unit 3) – 5 marks

For each of the following subsets of $\mathbf{R}^2$, state whether it is OPEN or NOT OPEN in $\mathbf{R}^2$ with the Euclidean metric.

$$H_1 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 + x_2 \neq 0\}$$
$$H_2 = H_1 \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}$$
$$H_3 = H_1 \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_2 < 0\}$$
$$H_4 = H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_1 \geq 0\}$$
$$H_5 = H_1 \cap \{(x_1, x_2) \in \mathbf{R}^2 : x_2 < 0\}$$ [5]