

Questions 1 to 4 are on Metric and Topological Spaces.
Questions 5 to 9 are on Geometric Topology.

Metric and Topological Spaces

Question 1 - 13 marks

For each positive integer n , let

$$A_n = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 > \frac{1}{n}x_1 \right\}.$$

(i) Sketch the subsets A_1, A_2 of \mathbb{R}^2 . [2]

(ii) Prove that if $x = (x_1, x_2)$ and $x_1 > 0, x_2 > 0$, then there exists a positive integer m such that $x \in A_m$. [4]

(iii) Prove that

$$\bigcup_{n=1}^{\infty} A_n = B \cup C,$$

where

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 > x_1\}$$

and

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}. \quad [7]$$

Question 2 - 12 marks

Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x_1, x_2) = |x_1| |x_2 - 1|, \quad (x_1, x_2) \in \mathbb{R}^2.$$

(i) Write down the following sets:

(a) $f(\mathbb{R}^2)$;

(b) $f([-1, 1] \times [2, 3])$;

(c) $f^{-1}(\{0\})$. [6]

(ii) Sketch the following subsets of \mathbb{R}^2 :

(a) $f^{-1}(\{1\})$;

(b) $f^{-1}([0, 1])$. [6]

Question 3 - 12 marks

Let $f: \mathbb{R} \rightarrow \mathbb{R}^2$ be the function given by

$$f(x) = (2 + |x|, 2 - |x|), \quad x \in \mathbb{R}.$$

(i) Sketch the subset $f(\mathbb{R})$ of \mathbb{R}^2 . [2]

(ii) Show that

$$f^{-1}(\{(3, 1)\}) = \{-1, 1\}. \quad [2]$$

(iii) Determine the following subsets of \mathbb{R} :

(a) $f^{-1}([2, 5] \times [1, 2])$;

(b) $f^{-1}([2, 5] \times [-2, 2])$. [8]