The easy way is to note that $G = \Gamma(K:\mathbb{Z}_3) = \langle\phi\rangle$ where $\phi$ is the Frobenius map (cf. Result K.4.2 (HB, p45)). $\phi(\alpha) = \alpha^3$ by def^n of Frob. and so $\phi^2(\alpha) = \phi(\alpha^3) = \alpha^9$. Now $\alpha^4+\alpha+2=0$ so $\alpha^4 = 2\alpha+1$, so $\alpha^8 = (2\alpha+1)^2 = \alpha^2+\alpha+1$ and so $\alpha^9 = \alpha\alpha^8 = \alpha(\alpha^2+\alpha+1) = \alpha^3+\alpha^2+\alpha$. So can take $\sigma = \phi^2$. This is a sensible way to proceed as the only $\mathbb{Z}_3$-autos of $K$ are $1, \phi, \phi^2, \phi^3$

$(2\alpha+1)(\alpha^2(2\alpha+1) + 2\alpha+1 + 2\alpha^3+\alpha^2+\alpha+1) + \alpha^3+\alpha^2+\alpha+2$

$= (2\alpha+1)(\alpha^3+2\alpha^2+2) + \alpha^3+\alpha^2+\alpha+2$

$= 2\alpha^4 + 2\alpha^3 + 2\alpha^2 + \alpha + 2 + \alpha^3 + \alpha^2 + \alpha + 2$

$= 2\alpha^4 + 2\alpha + 4 = 2(\alpha^4+\alpha+2) = 2m(\alpha) = 0.$

So both $\alpha$ and $\alpha^3+\alpha^2+\alpha$ are zeros of $m \Rightarrow$ there is an automorphism $\sigma$ that maps $\alpha$ to $\alpha^3+\alpha^2+\alpha$

(v) A general element of $K$ is

$k = a\alpha^3 + b\alpha^2 + c\alpha + d, \quad a,b,c,d \in \mathbb{Z}_3$

$\sigma(k) = \sigma(a)\sigma(\alpha^3) + \sigma(b)\sigma(\alpha^2) + \sigma(c)\sigma(\alpha) + \sigma(d)$

$= a(\sigma(\alpha))^3 + b(\sigma(\alpha))^2 + c\,\sigma(\alpha) + d$

$= a\alpha^3(\alpha^2+\alpha+1)^3 + b\alpha^2(\alpha^2+\alpha+1)^2 + c(\alpha^3+\alpha^2+\alpha) + d$

$= a\alpha^3(\alpha^4+\alpha^2+1+2\alpha^3+2\alpha^2+2\alpha)(\alpha^2+\alpha+1)$

$+ b\alpha^2(\alpha^4+\alpha^2+1+2\alpha^3+2\alpha^2+2\alpha) + c(\alpha^3+\alpha^2+\alpha) + d$

$= a\alpha^3(2\alpha+1+2\alpha^3+2\alpha+1)(\alpha^2+\alpha+1) + b\alpha^2(2\alpha+1+2\alpha^3+2\alpha+1)$

$+ c(\alpha^3+\alpha^2+\alpha) + d$

$= a\alpha^3(2\alpha^3+\alpha+2)(\alpha^2+\alpha+1) + b\alpha^2(2\alpha^3+\alpha+2)$

$+ c(\alpha^3+\alpha^2+\alpha) + d$

$= a(2\alpha^6+\alpha^4+2\alpha^3)(\alpha^2+\alpha+1) + b(2\alpha^5+\alpha^3+2\alpha^2)$

$+ c(\alpha^3+\alpha^2+\alpha) + d$

$= a(2\alpha^2(2\alpha+1) + 2\alpha+1 + 2\alpha^3)(\alpha^2+\alpha+1) + b(2\alpha(2\alpha+1) + \alpha^3 + 2\alpha^2)$

$+ c(\alpha^3+\alpha^2+\alpha) + d$

$= a(2\alpha^2+2\alpha+1)(\alpha^2+\alpha+1) + b(\alpha^3+2\alpha) + c(\alpha^3+\alpha^2+\alpha) + d$

$= a(2\alpha^4+\alpha^3+2\alpha^2+1)$

$+ b(\alpha^3+2\alpha) + c(\alpha^3+\alpha^2+\alpha) + d$

$= a(2(2\alpha+1)+\alpha^3+2\alpha^2+1) + b(\alpha^3+2\alpha) + c(\alpha^3+\alpha^2+\alpha) + d$

$= a(\alpha^3+2\alpha^2+\alpha) + b(\alpha^3+2\alpha) + c(\alpha^3+\alpha^2+\alpha) + d$

$= \alpha^3(a+b+c) + \alpha^2(2a+c) + \alpha(a+2b+c) + d$

Fixed field is the soln to

$a+b+c = a$
$2a+c = b$
$a+2b+c = c$
$d = d$

Rewrite them:

$b+c = 0$ (1)
$2a-b+c = 0$ (2)
$a+2b = 0$ (3)

(2) $- 2\times$(3) $\quad -5b+c = 0$
$\quad b+c = 0$ which is eqtn (1)

hence the system reduces to

$b+c = 0$ (4)
$2a-b+c = 0$ (5)

in which 1 parameter is arbitrary

Make $c$ the parameter

This is very laborious. Alternatively, to avoid calculations use Thm to the Fund. Thm of Galois Thy. By Result K.4.6 (HB p45) the Fund. Thm applies to the Galois extn $K:\mathbb{Z}_3$. By the Fund. Thm $[K^{\langle\sigma\rangle}:\mathbb{Z}_3] = 4/|\langle\sigma\rangle| = 4/2 = 2$ (since $\sigma^2 = \phi^4 = 1$) so $|K^{\langle\sigma\rangle}| = 3^2 = 9$.