$\cos\frac{2\pi}{7} = \frac{1}{2}(e^{i\frac{2\pi}{7}} + e^{-i\frac{2\pi}{7}})$ ③

$\therefore \cos\frac{2\pi}{7} \in \Sigma$ ∈ Σ for this reason, not because it is the real part of an elt of Σ [e.g. $\{|z|=1\} \cap \mathbb{R} = \{\pm 1\}$, not $[-1,1]$]

but $\cos\frac{2\pi}{7}$ is real $\therefore$ is a member of $\mathbb{R}$

3 so $\cos\frac{2\pi}{7} \in \Sigma \cap \mathbb{R}$, but it is not

4 rational ($\cos\frac{2\pi}{7} \notin \mathbb{Q}$) so $\Sigma \cap \mathbb{R} \neq \mathbb{Q}$

lose 1 mark A pure length l is constructible from P_0 if & only if the pt. $(l,0)$ is constructible from P_0.

(b) $t^7 - 1 = (t-1)(t^6+t^5+t^4+t^3+t^2+t+1)$; the polynomial of degree 6 is irreducible (Put $t = 1+u$ and use Eisenstein's criterion with $p = 7$)

$[\Sigma : \mathbb{Q}] = 6$

$\therefore [\Sigma : \Sigma\cap\mathbb{R}][\Sigma\cap\mathbb{R} : \mathbb{Q}] = 6$

For any line of length l in $\Sigma\cap\mathbb{R}$ to be constructible from P_0, the degree of the minimum polynomial of which l is a zero, over $\mathbb{Q}$ must divide $[\Sigma\cap\mathbb{R} : \mathbb{Q}]$

$\therefore \partial m \mid [\Sigma\cap\mathbb{R} : \mathbb{Q}]$

$\Rightarrow \partial m \leq 6 = 2\times 3$

10/16 (thanks) But $[\Sigma : \Sigma\cap\mathbb{R}] > 1$ since $\Sigma \notin \mathbb{R}$

$\partial m = 1, 2, 3$ but not 6.

This proves the converse of 'if' loses marks If $\partial m = 3$ then the minimum polynomial of l is not a power of 2 $\Rightarrow l$ not constructible $\Rightarrow \partial m = 1$ or 2 ≤ 2 as required

If $\partial m = 1$ then $l \in \mathbb{Q}$ and so $(l,0)$ is constructible from P_0 by Result 13.2.2 (HB p42) if $\partial m = 2$ then $[\mathbb{Q}(l):\mathbb{Q}] = 2$ so $(l,0)$ is constructible from P_0 by Result 13.2.4 (HB p42)

d i) $p(t) = t^5 - 10t^2 + 2$. Put $q = 2$ then by Eisenstein's criterion p is irreducible.

$p(-3) = -243 - 10\times 9 + 2 = -331 < 0$

$p(0) = 0 - 0 + 2 = 2 > 0$

$p(1) = 1^5 - 10\times 1^2 + 2 = -7 < 0$, $p(3) = 243 - 90 + 2 > 0$

As $t \to -\infty$ $p(t) \to -\infty$

As $t \to \infty$ $p(t) \to \infty$

It looks as though p has only two non real zeros, but we must be sure.

$p'(t) = 5t^4 - 20t = 5t(t^3 - 4)$

$p'(t) = 0$ at $t = 0$ and $t = \sqrt[3]{4}$ so p has two real turning points and 3 real zeros (shown on the opposite graph of $p(t)$). $p(t)$ is an irreducible polynomial of degree 5 over a field of characteristic zero $\therefore$ has five distinct zeros. Since three zeros are real, two are complex (conjugates) Result 18.3.2 (HB p35)

hence the permutation group