On the zeros of p contain a five cycle and a 2 cycle, so is S_5, by Result 14.B.4 (HB, p43)

13/13 S_5 is not soluble so $p(t)$ is not soluble by radicals. by Result 14.B.1 (HB, p43)

[illegible] [illegible] – they firmly base your arguments

ii) $g(t) = t^8 - 2$

The zeros of g are

$\alpha, \alpha\omega, \alpha\omega^2, \alpha\omega^3, \alpha\omega^4, \alpha\omega^5, \alpha\omega^6, \alpha\omega^7$ so g splits in $\mathbb{Q}(\alpha, \omega)$

where $\alpha = \sqrt[8]{2}$, $\omega = e^{\pi i/4}$. Call Σ the splitting field

✓ $\alpha, \alpha\omega \in \Sigma \Rightarrow \frac{\alpha\omega}{\alpha} = \omega \in \Sigma$ so Σ contains $\mathbb{Q}(\alpha, \omega)$

since g splits in $\mathbb{Q}(\alpha, \omega)$, it follows that $\mathbb{Q}(\alpha, \omega)$ is a splitting field of g over $\mathbb{Q}$

4/6 $\therefore \alpha, \omega \in \Sigma$ and since we can express any zero of $g(t)$ in the form $\alpha\omega^i$ $i = 0$ to 7, we can say $\Sigma = \mathbb{Q}(\alpha, \omega)$

(see general def^n HB p93)

3/4 b) $\alpha^8 = 2 \in \mathbb{Q}$ and $\varepsilon^4 = -1 \in \mathbb{Q}(\varepsilon)$ $\therefore \Sigma : \mathbb{Q}$ is a radical field extension (Stewart p128)

c) Yes. Use the defn on Stewart p128.

1/1 ✓ $\Sigma : \mathbb{Q}$ is a radical field extension $\therefore$ $g(t)$ is soluble by radicals.

iii) a) $\varepsilon = e^{\pi i/4}$ has minimum polynomial

NB it is irreducible over $\mathbb{Q}$ by consider $h(t+1)$ & using Eisenstein at $p = 2$.

$h(t) = t^4+1$ over $\mathbb{Q}$. This polynomial has zeros $e^{\pi i/4}, e^{-\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}$

$e^{\pi i/4} \in \mathbb{Q}(\varepsilon)$

$e^{3\pi i/4} = (e^{\pi i/4})^3 \in \mathbb{Q}(\varepsilon)$

$e^{5\pi i/4} = (e^{\pi i/4})^5 \in \mathbb{Q}(\varepsilon)$

$e^{-\pi i/4} = (e^{\pi i/4})^{-1} \in \mathbb{Q}(\varepsilon)$

$\therefore \mathbb{Q}(\varepsilon)$ contains all the zeros of t^4+1 ie it is a splitting field for t^4+1, and

5/6 a normal field extension of $\mathbb{Q}$ $\therefore \Gamma(\Sigma : \mathbb{Q}(\varepsilon))$ is a normal subgroup of G, by the fundamental theorem of Galois theory. ✓

b) Any $\mathbb{Q}(\varepsilon)$ automorphism of Σ leaves ε fixed. Thus if σ_j, σ_k are two elements of $\Gamma(\Sigma : \mathbb{Q}(\varepsilon))$.

(where $\sigma_j : \alpha \mapsto \alpha\varepsilon^j$?)

$\sigma_k\sigma_j(\alpha) = \sigma_k(\alpha\varepsilon^j) = \alpha\varepsilon^{jk} = \sigma_j(\alpha\varepsilon^k) = \sigma_j\sigma_k(\alpha)$

5/6 ✓ Since $\Gamma(\Sigma : \mathbb{Q}(\varepsilon))$ is determined solely by its effect on α, $\Gamma(\Sigma : \mathbb{Q}(\varepsilon))$ is abelian.