9

For each case, there exists $x, y \in K$ such that $y - x \in M_1$, $y - x \notin M_2$
$\Rightarrow M_1 \neq M_2$

iii) By i)a) either $1 \in M'$ or $-1 \in M'$
If $-1 \in M'$ then $(-1)(-1) = 1 \in M'$. In either case $1 \in M'$.
$1 + 1 = 2 \in M'$. Suppose $k \in M'$
Then $1 + k = k + 1 \in M'$.
Also 0 or -0 (both equal 0) $\in M'$
$\Rightarrow 0 \in M'$
$\Rightarrow$ Every integer in M is in M'

There [illegible] integers corresponding to [illegible] integers are properties of K not of any particular ordering. Suppose [illegible] for some $n \in \mathbb{Z}$ with $n \in M'$. Then $n+1 \in M'$ by i) a) above. Since $1 \in M'$, for all n, $n \in M'$. Also $0 \in M$, $0 \in M'$, so every integer in M is also in M'.

$1 \in M'$ and so $1 \geq' 0$
$1 - 1 \geq' 0 - 1 = -1$
so $0 \geq' -1$
so $-1 \notin M'$ (If $x \geq' 0$ then $0 \geq' -x$
Similarly so only non negative integers are $\in$ in M').

The field operations $+, \div, \times, -$ are the same in M and M': so
$M = M'$

Then from ii) $\leq' \cong \leq$ so all the ordering relations on $\mathbb{Q}$ which make it an ordered field are isomorphic.

B) i) $|\mathbb{Q}(\sqrt[n]{a}, \sqrt[n]{b}) : \mathbb{Q}| = |\mathbb{Q}(\sqrt[n]{a}, \sqrt[n]{b}) : \mathbb{Q}(\sqrt[n]{a})||\mathbb{Q}(\sqrt[n]{a}) : \mathbb{Q}|$
$\leq n \times n = n^2$
so $\mathbb{Q}(\sqrt[n]{a}, \sqrt[n]{b}) : \mathbb{Q}$ is a finite algebraic extn.
$\mathbb{Q}$ is an infinite field $\Rightarrow \sqrt[n]{a}$ separable
Then by Th. 16.3 $\mathbb{Q}(\sqrt[n]{a}, \sqrt[n]{b}) : \mathbb{Q}$ is a simple extension.

ii) $(\alpha + \beta)^3 + \alpha + \beta = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 + \alpha + \beta$
α a zero of $t^2 + 2 \Rightarrow \alpha^2 = -2 \equiv 3 \pmod 5$
β a zero of $t^2 + 3 \Rightarrow \beta^2 = -3 \equiv 2 \pmod 5$

$(\alpha + \beta)^3 + \alpha + \beta = 3\alpha + 9\beta + 6\alpha + 2\beta + \alpha + \beta$
$= 10\alpha + 12\beta$
$= 2\beta$

$(\alpha + \beta)^{-1} = \dfrac{1}{\alpha + \beta} = \dfrac{1}{\alpha + \beta} \times \dfrac{\alpha - \beta}{\alpha - \beta} = \dfrac{\alpha - \beta}{\alpha^2 - \beta^2}$