On this assignment you should answer only TWO questions; both of these will be used for assessment purposes.

Question 1 (Unit 13) – 50 marks

Let O and A be two given points whose coordinates are taken as being $(0,0)$ and $(1,0)$, respectively, and let P_0 be the subset $\{(0,0),(1,0)\}$ of $\mathbf{R}^2$.

(i) Determine whether or not each of the following may be constructed from P_0 using just straight-edge and compasses. Justify your answer in each case.

(a) An equilateral triangle of area 3. [9]

(b) A regular polygon with 3084 sides. [7]

(c) An angle of 10 degrees. [7]

(d) The acute angle $\frac{3}{4}\alpha$ where $\cos\alpha = \frac{3}{4}$. [7]

(ii) The subfield Σ of $\mathbf{C}$ is the splitting field for the polynomial $t^7 - 1$ over $\mathbf{Q}$.

(a) Use the fact that $\cos(2\pi/7)$ is irrational to show that $\Sigma \cap \mathbf{R} \neq \mathbf{Q}$. [4]

(b) Prove that a line of length ℓ in $\Sigma \cap \mathbf{R}$ can be constructed from P_0 using just straight-edge and compasses if and only if the degree of the minimum polynomial m of ℓ over $\mathbf{Q}$ is less than or equal to 2. [16]

Question 2 (Unit 14) – 50 marks

(i) Show that the polynomial $t^5 - 10t^2 + 2$ over the field $\mathbf{Q}$ is not soluble by radicals. [15]

(ii) The polynomial g over the field $\mathbf{Q}$ is given by $g(t) = t^8 - 2$.

(a) Show that $\Sigma = \mathbf{Q}(\alpha, \varepsilon)$ is a splitting field for g over $\mathbf{Q}$, where $\alpha = \sqrt[8]{2}$ and $\varepsilon = e^{i\pi/4}$. [6]

(b) Show that $\Sigma : \mathbf{Q}$ is a radical field extension. [4]

(c) Hence show that g is soluble by radicals. [1]

(iii) Consider now the Galois group G of $g(t) = t^8 - 2$ over $\mathbf{Q}$. From part (ii)(c) and Theorem 14.6, Result 14.3.1 (HB p. 43), we see that G is a soluble group. This part of the question asks you to show that G is not only soluble but also metabelian, as follows.

(a) Show that $\Gamma(\Sigma : \mathbf{Q}(\varepsilon))$ is a normal subgroup of G.
[Hint: Show that $\mathbf{Q}(\varepsilon)$ is a normal field extension of $\mathbf{Q}$.] [6]

(b) Show that $\Gamma(\Sigma : \mathbf{Q}(\varepsilon))$ is abelian. [6]

(c) Show that $\Gamma(\mathbf{Q}(\varepsilon) : \mathbf{Q})$ is abelian.
[Hint: Consider $[\mathbf{Q}(\varepsilon) : \mathbf{Q}]$, the degree of the minimum polynomial of ε over $\mathbf{Q}$.] [6]

(d) Hence show that G is metabelian. [6]

$1 = 3t - 4t^3$

$4t^3 - 3t + 1 = 0$

$\frac{1}{2} - \frac{3}{2} + 1 =$

$$\begin{array}{r|l} & 2t^2 + t + 1 \\ 2t - 1 & 4t^3 + 0t^2 - 3t + 1 \\ & 4t^3 - 2t^2 \\ & \quad 2t^2 - 3t \\ & \quad 2t^2 - t \\ & \qquad 2t + 1 \end{array}$$

$t = \frac{-1 \pm \sqrt{1-8}}{4}$