Question 3 (Unit 15) – 50 marks

Let K be the field of order 16 and let m be the polynomial in $\mathbf{Z}_2[t]$ defined by

$$m(t) = t^4 + t + 1.$$

Could be expressed as product of two quadratics

(i) Prove that m is irreducible over $\mathbf{Z}_2$. [8]

(ii) Prove that there is an element α in K such that α has a minimum polynomial m over $\mathbf{Z}_2$ and $K = \mathbf{Z}_2(\alpha)$. [16]

(iii) Show that there is an automorphism σ of K such that $\sigma(\alpha) = \alpha + 1$. [10]

(iv) Show that the field $\langle\sigma\rangle^\dagger$ has 4 elements. [16]

Question 4 (Unit 16) – 50 marks

(i) Consider the following statements about ordered fields. For each one, say whether the statement is true or false, justifying your answer with either a short proof or a counter-example.

Statement (a) If K is an ordered field and $k \in K$, then $k^2 \geq 0$. [2]

Statement (b) If K is an ordered field, then K is infinite. [5]

Statement (c) If K is an ordered field, then every field L that is a finite extension of K is ordered. [5]

Statement (d) If K is an ordered field, $x \in K$ and $x \geq 0$, then $x = k^2$ for some $k \in K$. [4]

Statement (e) If K is an ordered field, then every polynomial of odd degree in $K[t]$ has a zero in K. [4]

Statement (f) If K is an ordered field and $P = \{k \in K : 0 \leq k, 0 \neq k\}$, then $x, y \in P$ implies $xy \in P$. [6]

(ii) (a) Show that if an algebraic extension of a field K is simple, then it must also be finite. [4]

Consider now the field $B = \mathbf{A} \cap \mathbf{R}$, where $\mathbf{A}$ is the field of algebraic numbers.

(b) Show that $B : \mathbf{Q}$ is an algebraic extension but B is not algebraically closed. [6]

(c) Show that for each integer $n > 0$, $\mathbf{Q}(\sqrt[n]{2})$ is a subfield of B such that $[\mathbf{Q}(\sqrt[n]{2}) : \mathbf{Q}] = n$. [4]

(d) Deduce that $B : \mathbf{Q}$ is not a simple extension. [6]

(e) Show that for any $a, b \in B$, $\mathbf{Q}(a, b) : \mathbf{Q}$ is a simple extension. [4]

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