Question 3 (Unit 11) – 25 marks

The field extension $L : K$ is finite, normal and separable, and M and N are subfields of L containing K. Suppose that the extensions $M : K$ and $N : K$ are normal, that $M \cap N = K$, and that the only subfield of L containing both M and N is L itself.

The Galois groups $\Gamma(L : K)$, $\Gamma(L : M)$ and $\Gamma(L : N)$ are denoted by G, M^* and N^*, respectively.

(i) Prove that M^*N^* is a subgroup of G. [3]

(ii) Prove that $(M^*N^*)^\dagger$, the fixed subfield of M^*N^*, is $K = M \cap N$, and deduce that $M^*N^* = G$. [7]

(iii) Prove that $(M^* \cap N^*)^\dagger$, the fixed subfield of $M^* \cap N^*$, is L, and deduce that $M^* \cap N^* = 1$. [7]

(iv) Using the second isomorphism theorem, or otherwise, prove that the Galois groups $\Gamma(M : K)$ and $\Gamma(L : N)$ are isomorphic. [5]

(v) Prove that G is isomorphic to the direct product $\Gamma(M : K) \times \Gamma(N : K)$. [3]

Question 4 (Unit 12) – 25 marks

A group is said to be supersoluble if it has a finite series of subgroups

$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$$

such that for $i = 0, 1, 2, \ldots, n-1$, (a) $G_i \lhd G$ and (b) G_{i+1}/G_i is cyclic.

(i) Prove that if a group is supersoluble, then it is soluble. [3]

(ii) The three non-abelian groups $\mathbf{S}_3$, $\mathbf{D}_8$ and $\mathbf{S}_4$ are all soluble. Show that two of these groups are supersoluble, and one is not. [8]

(iii) Prove that every subgroup of a supersoluble group is soluble. [10]

(iv) Prove that $\mathbf{S}_n$ is not supersoluble for every $n \geq 4$. [4]