Let's try to construct a chain of such subgroups for S_4. S_4 has subgroups of the form $C_2, C_3, C_4, V, S_3, D_8, A_4$ and S_4

From the right the 1st link would have to be * $A_4 \triangleleft S_4 = S_4$ since S_3, D_8 etc are not normal in S_4 (there is a subgroup isomorphic to the Klein group, but the quotient group is not cyclic).

$A_4 = \{e, (123), (132), (234), (243), (124), (142), (134), (143), (12)(34), (13)(24), (14)(23)\}$

A_4 has a subgroup $V = \{e, (12)(34), (13)(24), (14)(23)\}$ normal in S_4, and no other (it has 4 cyclic groups order 3, not normal in S_4, and the quotient groups are also not cyclic, and no subgroup order 6, which could only be S_3 which is not normal in S_4, so we have to take V as above. V is not cyclic so we have to choose a subgroup consisting of the identity and an element order 2. But all the elements in the group V given are of the same form, hence conjugate in S_4. There are 3 conjugate subgroups order 2, and these are hence not normal in S_4 hence there is no chain.

Q2 ✓

You can shorten this a little by realising that the only normal subgps of S_4 are $\{e\}$, V, A_4, S_4 Result 6.7 (HB, p.30)

iii) Suppose we have a supersoluble group. Then we can write down a chain of subgroups satisfying a) and b)

$1 = G_0 \leq G_1 \leq G_2 \ldots\ldots G_{n-1} \leq G_n = G$

Suppose G' is any subgroup of G. We form the chain

$1 = G_0 \cap G' \leq G_1 \cap G' \leq G_2 \cap G' \ldots G_{n-1} \cap G' \leq G_n \cap G' = G \cap G' = G'$

$G_i \triangleleft G \Rightarrow^{*} G_i \cap G' \triangleleft G' \Rightarrow G_i \cap G' \triangleleft G_{i+1} \cap G'$ * Result 6.3,5 (HB p.29)

so 1st condition for solubility satisfied (and, incidently, for supersolubility *)

For this you need $G_i \cap G_{i+1}' \triangleleft G'$

[illegible]