As a matter of notation you would find the argument easier to follow if instead of G' you wrote H for the subgp. This would spare ... & you could write $H_i = G_i \cap H$.

(1)

The second condition requires we prove $\frac{G_{i+1} \cap G'}{G_i \cap G'}$ is Abelian. If we can prove it is cyclic, then we know it is Abelian.

2nd ... for supersolubility

$G_i \subseteq G_{i+1} \Rightarrow G_i \cap G' = G_i \cap (G_{i+1} \cap G')$

From the 2nd Isomorphism Theorem with $N = G_i$ and $H = G_{i+1} \cap G'$

$$\frac{G_i(G_{i+1} \cap G')}{G_i} \simeq \frac{G_{i+1} \cap G'}{G_i \cap (G_{i+1} \cap G')}$$

But $\frac{G_{i+1} \cap G'}{G_i \cap G'} = \frac{G_{i+1} \cap G'}{G_i \cap (G_{i+1} \cap G')} \simeq \frac{G_i(G_{i+1} \cap G')}{G_i} \subseteq \frac{G_{i+1}}{G_i}$

and $\frac{G_{i+1}}{G_i}$ is cyclic

Every subgroup of a cyclic group is cyclic, hence Abelian. ... proved for supersolubility

Condition b) is satisfied $\therefore$ every subgroup of a supersoluble group is supersoluble.

$\frac{8}{10}$? ... as you gave no ... of supersolubility of G' ... different ... (although you didn't really ...)

1

iv) We use part iii), and the technique of intersecting the chain, then it becomes a proof by contradiction.

Suppose $n \geq 4$ and S_n is supersoluble. then there exists a finite chain of subgroups,

$1 = G_0 \subseteq G_1 \subseteq \ldots\ldots \subseteq G_n = S_n$

Such that

a) $G_i \triangleleft G$

b) G_{i+1}/G_i is cyclic

Take the intersection of each G_i with S_4. Note $S_4 \subseteq S_n$ $n \geq 4$

$1 = G_0 \cap S_4 \subseteq G_1 \cap S_4 \ldots\ldots \subseteq G_n \cap S_4 \subseteq S_n$

Then by iii) S_4 is supersoluble, a contradiction $\therefore S_n$ not supersoluble.

$\frac{4}{4}$ good