(ii) Let $K$ be a field, and let $g$ be an inseparable, irreducible, monic cubic polynomial in $K[t]$. Let $L$ be a field containing $K$ in which $g$ has a zero $\gamma$. Prove that:

(a) the field $K$ has characteristic 3; [3]

(b) $g(t) = t^3 - \gamma^3$; [3]

(c) all the zeros of $g$ are the same; [3]

(d) the field $L$ contains a splitting field for $g$ over $K$. [3]

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## TMA M433 03 — Cut-off date 28 July 1999

All four questions on this assignment will be used for assessment purposes.

### Question 1 (Unit 9) – 25 marks

Let $L$ be a field with automorphism group $G$, where $G = \{1, \sigma, \tau, \sigma\tau\}$ with

$$\sigma^2 = \tau^2 = 1, \quad \sigma\tau = \tau\sigma,$$

i.e. $G \cong \mathbf{V}$, the Klein Four-group described in Solution 2.1.7 of *Unit 2*.

Define $S = \{1, \sigma\}$ and $T = \{1, \tau\}$. You may assume that $S$ and $T$ are subgroups of $G$.

(i) Write down a non-trivial, proper subgroup of $G$ other than $S$ and $T$. [1]

(ii) Prove that there is some element $x \in L$ such that $\sigma(x) \neq x$. [2]

(iii) By using Theorem 9.4, Result 9.1.3 (HB p. 36), or otherwise, prove that $S^\dagger \neq G^\dagger$. [5]

(iv) By considering the effect of $\sigma$ and $\tau$ on an element of $S^\dagger \cap T^\dagger$, prove that $S^\dagger \cap T^\dagger = G^\dagger$. [6]

(v) Deduce that $S^\dagger \neq T^\dagger$. [2]

(vi) Prove that $L$ has at least 5 distinct subfields. [5]

(vii) (a) Choose a field $L$ from *Unit 7*, with automorphism group $\mathbf{V}$, such that $L$ has exactly five distinct subfields.

(b) Show that $\mathbf{Q}(\sqrt[4]{3}, \sqrt{2})$, which you may assume to have automorphism group $\mathbf{V}$, has more than five distinct subfields. [4]

### Question 2 (Unit 10) – 25 marks

Let $\omega$ be a non-real cube root of 1 in $\mathbf{C}$, let $\gamma$ be the positive real sixth root of 5, and let $L$ be the splitting field in $\mathbf{C}$ for the polynomial $t^6 - 5$ in $\mathbf{Q}[t]$.

(i) Show that $\mathbf{Q}(\omega)$ contains all the zeros of $t^6 - 1$. [3]

(ii) Show that $L = \mathbf{Q}(\gamma, \omega)$. [4]

(iii) Show that $[L : \mathbf{Q}] = 12$. [5]

(iv) Find $|\Gamma(L : \mathbf{Q})|$. [2]

(v) Explain why each element of $\Gamma(L : \mathbf{Q})$ is completely determined by its effect on $\gamma$ and $\omega$. [3]

(vi) Show that the elements of $\Gamma(L : \mathbf{Q})$ are all the products

$$\sigma^i\tau^j, \quad i = 1, \ldots, 6, \; j = 0, 1,$$

where $\sigma : \gamma \longmapsto -\omega\gamma, \quad \sigma : \omega \longmapsto \omega,$

and $\tau : \gamma \longmapsto \gamma, \quad \tau : \omega \longmapsto \overline{\omega}.$ [8]