4) i) If a group is soluble it has a finite series of subgroups
$1 = G_0 \subseteq G_1 \ldots \subseteq G_{n-1} \subseteq G_n = G$
such that for $i = 0, 1, \ldots, n-1$,
a) $G_i \triangleleft G_{i+1}$
b) G_{i+1}/G_i is Abelian

If $G_i \triangleleft G$ then $G_i \triangleleft G_{i+1}$ (since $G_i \subseteq G$, so if $g_i \in G_i$, $g \in G$ then $g g_i g^{-1} \in G_i$ for all g_i, g, but if $g_{i+1} \in G_{i+1}$, $g_{i+1} \in G$ $\therefore$ $g_{i+1} g_i g_{i+1}^{-1} \in G_i$)

G_{i+1}/G_i is cyclic, it is Abelian

$\frac{3}{3}$ ✓ $\therefore$ supersolubility $\Rightarrow$ solubility

ii) S_3 : $\{e\} \subseteq A_3 \subseteq S_3$

a) $\{e\} \triangleleft S_3$, $A_3 \triangleleft S_3$ $\because$ A_3 index 2
b) $A_3/\{e\} \simeq C_3$, $S_3/A_3 \simeq C_2$

$\frac{3}{2}$ ✓ $\therefore$ S_3 is supersoluble.

$D_8 : \{e\} \subseteq \{e, r^2\} \subseteq \{e, r, r^2, r^3\} \subseteq D_8$

a) $\{e\} \triangleleft D_8$
$\{e, r^2\} \triangleleft D_8$;

[A general element of D_8 is $r^i q^j$
$r^i q^j \{e, r^2\} (r^i q^j)^{-1}$
$= r^i q^j \{e, r^2\} q^{j} r^{-i}$ $\quad (q^j = q^{-j})$
$= \{e, r^i q^j r^2 q^j r^{-i}\}$
$= \{e, r^i q^j q^j r^2 r^{-i}\}$
$= \{e, r^i r^2 r^{-i}\} = \{e, r^2\}$]

$\{e, r, r^2, r^3\} \triangleleft D_8$ (since it has index 2 in D_8)
$\therefore$ Condition a) satisfied.

b) $\frac{D_8}{\{e, r, r^2, r^3\}} \simeq \frac{\{e, r, r^2, r^3\}}{\{e, r^2\}} \simeq \frac{\{e, r^2\}}{\{e\}} \simeq C_2$
$\therefore$ condition b) satisfied.

$\frac{3}{2}$ ✓ (or could omit $\{e, r^2\}$)
so, S_3 & D_8 are supersoluble.