All four questions on this assignment will be used for assessment purposes.

Question 1 (Unit 9) – 25 marks

Throughout this question, f is an irreducible cubic polynomial in $\mathbf{Q}[t]$.

(i) Prove that if the complex number γ is a zero of f, then its complex conjugate $\overline{\gamma}$ is also a zero of f. *[4]*

Let the zeros of f in $\mathbf{C}$ be α, β, γ, and let $L = \mathbf{Q}(\alpha, \beta, \gamma)$. Let $\sigma : L \longrightarrow \mathbf{C}$ be defined by $\sigma(x) = \overline{x}$.

(ii) Prove that $\sigma(L) \subseteq L$ and that $L \subseteq \sigma(L)$. *[4]*

(iii) Hence, or otherwise, show that $\sigma : L \longrightarrow L$ is a $\mathbf{Q}$-automorphism of L. *[2]*

(iv) Show that if the zeros α, β, γ are *not* all real, then:

(a) $\Gamma(L : \mathbf{Q})$ has a subgroup of order 2; *[3]*

(b) there is a subfield K of L containing $\mathbf{Q}$ such that $[L : K] = 2$; *[3]*

(c) $[L : \mathbf{Q}]$ is divisible by 6; *[3]*

(d) $\mathbf{Q}(\alpha) : \mathbf{Q}$ is not a normal extension. *[6]*

Question 2 (Unit 10) – 25 marks

The field N is the normal closure of $L : \mathbf{Q}$ in $\mathbf{C}$, where $L = \mathbf{Q}(\alpha)$ and α is the real fifth root of 7.

(i) Find $[L : \mathbf{Q}]$ and $|\Gamma(L : \mathbf{Q})|$. *[5]*

(ii) Find the number of $\mathbf{Q}$-monomorphisms from

(a) L into $\mathbf{R}$;

(b) L into N;

(c) L into $\mathbf{C}$. *[4]*

(iii) Prove that $N = L(\beta)$, where $\beta = e^{2\pi i/5}$. *[4]*

(iv) Show that $[\mathbf{Q}(\beta) : \mathbf{Q}] = 4$, and hence that $[N : \mathbf{Q}]$ is divisible by 20. *[7]*

(v) Show that $[N : \mathbf{Q}(\beta)] \leq 5$, and hence that $[N : \mathbf{Q}] = 20$. *[5]*

Question 3 (Unit 11) – 25 marks

Let $\varepsilon = e^{2\pi i/8}$. You may assume that the minimum polynomial of ε over $\mathbf{Q}$ is $t^4 + 1$.

(i) Prove that $\mathbf{Q}(\varepsilon)$ is a splitting field for $t^4 + 1$ over $\mathbf{Q}$. *[3]*

Let G be the Galois group $\Gamma(\mathbf{Q}(\varepsilon) : \mathbf{Q})$.

(ii) Find all the members of G. *[5]*

(iii) List all the subgroups of G. *[4]*

(iv) How many subfields does $\mathbf{Q}(\varepsilon)$ have? Justify your answer. *[5]*

(v) Find all subfields of $\mathbf{Q}(\varepsilon)$, expressing your answers in the form $\mathbf{Q}(x)$, for suitable elements x of $\mathbf{Q}(\varepsilon)$. *[8]*