Question 4 (Unit 12) – 25 marks

The *soluble length*, $l(G)$, of a soluble group G is the *smallest* (non-negative) integer n such that G has a finite series of subgroups

$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$$

such that

$$(1)\ G_i \triangleleft G_{i+1} \quad \text{and} \quad (2)\ G_{i+1}/G_i \text{ is abelian,} \quad \text{for } i = 0, 1, \ldots, n-1.$$

(i) Find $l(\mathbf{C}_6)$ and $l(\mathbf{S}_3)$. [4]

(ii) A soluble group G has a proper non-trivial normal subgroup N. By considering Theorem 13.2 and its proof, or otherwise, prove that

(a) $l(N) \leq l(G)$, [5]

(b) $l(G) \leq l(N) + l(G/N)$. [6]

(iii) Give examples to show that amongst soluble groups G and proper non-trivial normal subgroups N:

(a) both cases $l(N) < l(G)$ and $l(N) = l(G)$ occur; [4]

(b) both cases $l(G) < l(N) + l(G/N)$ and $l(G) = l(N) + l(G/N)$ occur. [6]

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On this assignment you should answer only TWO questions; both of these will be used for assessment purposes.

Question 1 (Unit 13) – 50 marks

Determine whether or not each of the following may be constructed using just a straight-edge and compasses. Justify your answer in each case.

(i) A cube with volume seven times that of a given cube. [9]

(ii) An equilateral triangle equal in area to a given square. [9]

(iii) An angle of $\pi/170$ radians. [9]

(iv) An acute angle α such that $\cos(\alpha) = \frac{9}{16}$. [9]

(v) The point with coordinates $(0, x)$, where

(a) $x = \sqrt[4]{2}$, [7]

(b) $x = \sqrt[3]{1 + \sqrt{7}}$. [7]

The points $(0,0)$, $(1,0)$ and $(0,1)$ are given.

Question 2 (Unit 14) – 50 marks

(i) Show that the following polynomials are not soluble by radicals over $\mathbf{Q}$.

(a) $t^5 - 4t^2 + 2$ [15]

(b) $t^7 - 10t^5 + 15t + 5$ [20]

(Hint: Differentiate more than once if necessary.)

(ii) Find a quintic polynomial which is irreducible over $\mathbf{Q}$ but is soluble by radicals over $\mathbf{Q}$, fully justifying your answer. [15]