b) $G/(A_3 \times \{0\})$. (corresponding to G)

They are normal subgroups of G because G/N is abelian (isomorphic to the Klein group, which is abelian), and so all the subgroups of G/N are normal in G/N. By the correspondence theorem the corresponding subgroups of G are normal subgroups of G (they contain N)

$\frac{3}{3}$ ✓

c) $\{e\} \times G$ and $\{e\} \times \{0\}$ are normal in G (G is also normal in G).

$\frac{2}{2}$ ✓

3) i) $m(t) = t^3 + t + 1$

$m(0) = 0^3 + 0 + 1 = 1 \pmod 2$

$m(1) = 1^3 + 1 + 1 \equiv 1 \pmod 2$

$\frac{2}{3}$

$m(t)$ has no zeros in $\mathbb{Z}_2$ ∴ it is irreducible over $\mathbb{Z}_2$ i.e. no linear factors over $\mathbb{Z}_2$

ii) $m(\beta) = \beta^3 + \beta + 1 = 0$ since β is a zero of m in $\mathbb{F}_8$

$m(\beta^2) = (\beta^2)^3 + \beta^2 + 1 = \beta^6 + \beta^2 + 1$

$$\begin{array}{r|l} & \beta^3 + \beta + 1 \\ \hline \beta^3+\beta+1 & \beta^6 + 0\cdot\beta^5 + 0\,\beta^4 + 0\,\beta^3 + \beta^2 + 0\cdot\beta + 1 \\ & \beta^6 + \beta^4 + \beta^3 \\ \hline & + \beta^4 + \beta^3 \\ & \beta^4 + \beta^2 + \beta \\ \hline & \beta^3 + \beta + 1 \\ & \beta^3 + \beta + 1 \\ \hline \end{array}$$

$\frac{1}{4}$ ✓

so in $\mathbb{F}_8$ $m(\beta^2) = (\beta^3 + \beta + 1)^2 = 0^2 = 0$ and β^2 is also a zero of m.

as in $\mathbb{Z}_2$ $(\beta^3+\beta+1)^2 = \beta^6 + \beta^2 + 1 + 2(\beta^3\beta + \beta^3 \cdot 1 + \beta \cdot 1)$

$\beta^6 + \beta^2 + 1 = (\beta^3)^2 + (\beta^2) + 1$

iii) $m(a) = a^3 + a + 1$

$= a \cdot a \cdot a + a + 1$

$= a \cdot b + c$

$= c + c = 0$ ∴ a is a zero of m.

$\frac{2}{2}$ ✓

iv).