$a = a$
$a^2 = b$
$a^3 = a \cdot b = c$
$a^4 = a \cdot c = d$
$a^5 = a \cdot d = e$
✓ $a^6 = a \cdot e = f$
$a^7 = a \cdot f = 1$

iv) ✓ $\{F_8\} \setminus \{0\} \times 3$ is generated by a so $F_8 \subseteq \mathbb{Z}_2(a)$ (since $0 \in \mathbb{Z}_2$)

$\frac{2}{3}$ But $\mathbb{Z}_2 \subseteq F_8$ + $a \in F_8$ so $\mathbb{Z}_2(a) \subseteq F_8$.

v) $m(a) = 0$
From ii), since $m(a) = 0$, $m(a^2) = 0$ and applying again, $m(a^4) = 0$ and again, $m(a^8) = m(a) = 0$ (since $a^7 = 1$)
$\therefore$ m has at least three zeros,
$\frac{4}{4}$ ✓ but m is a polynomial of degree three so has at most three zeros $\therefore$ m has exactly three zeros (and $m(t) = (t-a)(t-b)(t-d)$).

vi) $F_8 = \mathbb{Z}_2(a)$, and the minimum polynomial of a over $\mathbb{Z}_2$ has order 3 and so $[F_8 : \mathbb{Z}_2] = \partial m = 3$

$\frac{3}{3}$ ✓
$\phi_1(a) = a$
$\phi_2(a) = b$
$\phi_2(a) = d$
} a, b, d are roots of m. Each ϕ permutes the roots.

vii) $\phi_1(a) = a$
$\phi_1(b) = \phi_1(a a) = \phi_1(a)\phi_1(a) = a \cdot a = b$
$\phi_1(c) = \phi_1(b) \cdot \phi_1(a) = ba = c$
$\phi_1(d) = \phi_1(c)\phi_1(a) = ca = d$
$\phi_1(e) = \phi_1(d)\phi_1(a) = da = e$
✓ $\phi_1(f) = \phi_1(ea) = \phi_1(e)\phi_1(a) = ea = f$
$\phi_1(0) = 0$
$\phi_1(1) = 1$

$\phi_2(a) = b$
$\phi_2(b) = \phi_2(a a) = \phi_2(a)\phi_2(a) = b \cdot b = d$
$\phi_2(c) = \phi_2(b \cdot a) = \phi_2(b)\phi_2(a) = d b = f$
$\phi_2(d) = \phi_2(c \cdot a) = \phi_2(c)\phi_2(a) = f \cdot b = a$
$\phi_2(e) = \phi_2(d \cdot a) = \phi_2(d)\phi_2(a) = a b = c$
✓ $\phi_2(f) = \phi_2(e \cdot a) = \phi_2(e)\phi_2(a) = cb = e$
$\phi_2(0) = 0$
$\phi_2(1) = 1$