All four questions on this assignment will be used for assessment purposes.

Question 1 (Unit 6) – 25 marks

This question concerns the set G of all functions $\mathbf{Z}_5 \longrightarrow \mathbf{Z}_5$ of the form

$$x \longmapsto ax + b \pmod 5), \quad \text{where } a, b \in \mathbf{Z}_5 \text{ and } a \neq 0.$$

You may assume that, with the group operation of composition of functions, these elements form a group.

Regarding the elements of $\mathbf{Z}_5$ as 0, 1, 2, 3 and 4, the elements of G are *permutations* of the symbols 0, 1, 2, 3 and 4. (This follows since the elements of G are invertible.) As is normal for permutations, we shall write them on the *right*.

Define the elements f, g and h in G as follows:

$$xf = x + 1,$$
$$xg = 4x,$$
$$xh = 2x.$$

(i) Express each of the permutations f, g and h as a product of disjoint cycles. [3]

(ii) Calculate the conjugate $h^{-1}fh$, and hence show that f is conjugate to each of f^2, f^3 and f^4. [4]

(iii) Show that $C_G(f)$, the centralizer in G of f, has order 5, i.e. $|C_G(f)| = 5$. Show also that f^G, the conjugacy class of G containing f, has order 4, i.e. $|f^G| = 4$. [4]

Let H be the subgroup $\langle h \rangle$ of G, and let N be the subgroup $\langle f \rangle$ of G.

(iv) Show that N is a normal subgroup of G, but H is not a normal subgroup. [6]

(v) Show that $G = HN$. [3]

(vi) Show that the quotient group G/N is isomorphic to $\mathbf{C}_4$. [5]

Question 2 (Unit 6) – 25 marks

(i) Let G be the dihedral group $\mathbf{D}_{24}$ of order 24, and let $R \subseteq G$ be the cyclic subgroup of G consisting of all the rotations in G. Let $a, b \in G$ be such that $\langle a \rangle = R$ and $b \notin R$. Let $H = \langle a^2 \rangle$ and $K = \langle a^6 \rangle$.

(a) Show that H and K are normal subgroups of G. [5]

(b) Find the order of each of the groups

$$G/H, \quad G/K, \quad H/K \quad \text{and} \quad (G/K)/(H/K).$$ [2]

(c) Write out all the elements (as cosets) of each of the quotient groups in part (i)(b), writing each element of G in the form

$$a^r b^s, \quad \text{where } 0 \le r < 12 \text{ and } 0 \le s < 2, \text{ wherever they occur.}$$ [4]

(d) Define an isomorphism $\phi : L \longrightarrow G/H$, where $L = (G/K)/(H/K)$, by giving the value of $\phi(x)$ for each $x \in L$.

Is ϕ the only isomorphism $L \longrightarrow G/H$? Justify your answer. [4]

(ii) Let $G = \mathbf{S}_3 \times \mathbf{C}_2$.

(a) Show that G has a normal subgroup N of order 3, and show that the quotient group G/N is not cyclic. [5]

(b) By considering G/N, or otherwise, find all the subgroups of G which contain N, and explain briefly why they are normal subgroups of G. [3]

(c) The group G has just two more normal subgroups. Write them down. [2]