**Question 4** (Unit 4) – 20 marks

Here $V$ is the vector space over $\mathbf{Q}$ consisting of all polynomials in $\mathbf{Q}[t]$ with degree at most 4; $W$ is the vector space over $\mathbf{Q}$ with basis $B = \{1, \alpha, \alpha^2, i, i\alpha, i\alpha^2\}$, where $i^2 = -1$ and $\alpha$ is the real cube root of 2. (You may assume that the members of $B$ are linearly independent over $\mathbf{Q}$.) Let $\phi : V \longrightarrow W$ be the function defined by

$$\phi : \sum_{r=0}^{4} a_r t^r \longmapsto \sum_{r=0}^{4} a_r i^r.$$

(i) Write down $\phi(t^2 + 3t + 2)$. [1]

(ii) Show that $\phi$ is a linear transformation. [3]

(iii) Write down an ordered basis $B'$ for $V$, and the matrix of $\phi$ with respect to $B'$ and $B$, with $B$ ordered as above. [3]

(iv) Find a basis for the image of $\phi$. [3]

(v) Write down $\dim(\mathrm{Im}(\phi))$, and hence find $\dim(\mathrm{Ker}(\phi))$. [3]

(vi) Find a basis for the kernel of $\phi$. [4]

For the final part of this question, you may assume that $W$ and $\mathrm{Im}(\phi)$ are fields.

(vii) Find the degree $[W : \mathrm{Im}(\phi)]$ of $W$ over $\mathrm{Im}(\phi)$, justifying your answer briefly. [3]

**Question 5** (Unit 5) – 20 marks

Let $K$ be the field $\mathbf{Q}(\sqrt{5})$ and let $L$ be the field

$$L = K(\sqrt{3}) = \mathbf{Q}(\sqrt{5}, \sqrt{3}).$$

(i) Find $[K : \mathbf{Q}]$, justifying your answer. [2]

(ii) Show that $\sqrt{3} \notin K$. [2]

(iii) Find $[L : \mathbf{Q}]$, justifying your answer. [4]

(iv) Prove that $\mathbf{Q}(\sqrt{5} + \sqrt{3}) = L = \mathbf{Q}(\sqrt{5}, \sqrt{3})$. [6]

(v) Find the minimum polynomial of $\sqrt{5} + \sqrt{3}$ over $\mathbf{Q}$, justifying your answer. [4] prove irreducible

(vi) Write down values of $\alpha \in \mathbf{Z}$, not equal to 5 or 3, such that since its monic of $\partial = 4$ it is minimum polyn. of $\sqrt{5}+\sqrt{3}$

(a) $[\mathbf{Q}(\sqrt{5} + \sqrt{\alpha}) : \mathbf{Q}] = 2$;

(b) $[\mathbf{Q}(\sqrt{5} + \sqrt{\alpha}) : \mathbf{Q}] = 4$. [2]

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## TMA M433 02 — Cut-off date 8 June 1995

All four questions on this assignment will be used for assessment purposes.

**Question 1** (Unit 6) – 25 marks

Let $G$ be the dihedral group $\mathbf{D}_{2n}$, of order $2n$, and let $R \subseteq G$ be the cyclic subgroup consisting of all the rotations in $G$. Let $a, b \in G$ be such that $\langle a \rangle = R$ and $b \notin R$.

(i) Write down the order of $b$ and the order of $R$. [2]

(ii) Find $b^{-1}ab$. [2]

(iii) Let $a^r$ be an element of $R$ which is in the centralizer of $b$ (i.e. $a^r b = b a^r$).

(a) Show that $a^r = a^{-r}$, and hence that $2r$ is divisible by $n$. [5]

(b) Hence show that there are at most two such elements $a^r$. [5]