(iv) For each of the following five values of n, find $Z(G)$, the centre of the dihedral group $\mathbf{D}_{2n}$:

$$n = 2, 4, 7, 103, 150.$$ [11]

Question 2 (Unit 6) – 25 marks

(i) Write down the centres of the groups $\mathbf{V}$ and $\mathbf{S}_3$. [4]

Let G be a group and let $Z(G)$ be its centre. We say that G is *central-by-abelian* if and only if the quotient group $G/Z(G)$ is abelian.

(ii) Let $H = Z(G)$ and let N be any normal subgroup of G. Prove that HN/N is a subgroup of the centre of G/N. [4]

(iii) Which, if any, of the groups $\mathbf{V}$, $\mathbf{S}_3$, $\mathbf{D}_8$ is central-by-abelian? Justify your answer. [7]

(iv) Prove that if a group G is central-by-abelian and N is a normal subgroup of G, then the quotient group G/N is also central-by-abelian. [10]

Question 3 (Unit 7) – 25 marks

The polynomial f in $\mathbf{Q}[t]$ defined by $f(t) = t^3 - 3t - 1$ has a zero α in $\mathbf{R}$.

(i) Show that f is irreducible in $\mathbf{Q}[t]$. [3]

(ii) Let $\beta = 2 - \alpha^2$. Show that β is a zero of f in $\mathbf{R}$. [3]

(iii) Show that $\beta \neq \alpha$. [2]

(iv) Write down $[\mathbf{Q}(\alpha) : \mathbf{Q}]$ and an expression for the general element of $\mathbf{Q}(\alpha)$ in terms of α and elements of $\mathbf{Q}$. [2]

(v) Let γ be the third zero of f in $\mathbf{C}$. By expanding $(t-\alpha)(t-\beta)(t-\gamma)$, show that $\alpha + \beta + \gamma = 0$. Hence find an expression for γ in terms of α, and show that f has three distinct roots, all in $\mathbf{R}$. [4]

(vi) Show that the order of the Galois group $G = \Gamma(\mathbf{Q}(\alpha) : \mathbf{Q})$ is 3, and write down the value of $\phi(\alpha)$ for each ϕ in G. [6]

(vii) Use your results from part (vi) to determine, for each $\phi \in G$, the image under ϕ of a *general* element of $\mathbf{Q}(\alpha)$, expressing your answers as polynomials in α of lowest possible degree. [5]

Question 4 (Unit 8) – 25 marks

(i) Show that $\mathbf{Q}(\sqrt{5}) : \mathbf{Q}$ is a normal extension. [2]

(ii) Show that $\mathbf{Q}(\sqrt[n]{5}) : \mathbf{Q}$ is not a normal extension for any integer value of n greater than or equal to 3. [5]

(iii) Let the complex number ζ be a primitive nth root of 1; that is,

$$\zeta^n = 1, \text{ but if } 0 < m < n \text{ then } \zeta^m \neq 1.$$

Prove that the extension $\mathbf{Q}(\zeta) : \mathbf{Q}$ is a normal extension. [7]

(iv) Let $L = \mathbf{Z}_2(x)$, where $\mathbf{Z}_2(x) : \mathbf{Z}_2$ is a simple transcendental extension, and let $K = \mathbf{Z}_2(x^2)$.

(a) Show that $K : \mathbf{Z}_2$ is a simple transcendental extension. [3]

(b) Show that the minimum polynomial of x over K is $t^2 - x^2$. [3]

(c) Hence show that $L : K$ is a simple algebraic extension but is not a separable extension. [5]