Question 14

(i) Which of the following terms are freely substitutable for x in the formula

$$\forall z(\exists y(x' + t) = y \;\&\; \forall x \forall t(t \cdot y) = x)?$$

(a) $(t + x)$

(b) $(\mathbf{0}' + z)$

(c) $(\mathbf{0}' + x)$

Answer YES or NO in each case. [2]

(ii) Let ϕ and ψ be formulas, where the variable x does not appear free in ψ, but might appear free in ϕ. Give formal proofs to show each of the following.

(a) $\neg \exists x \phi, \phi \vdash (\psi \;\&\; \neg \psi)$ [3]

(b) $\neg \exists x \phi \vdash \forall x \neg \phi$ [3]

(c) $\forall x \neg \phi \vdash \neg \exists x \phi$ [3]

[You may find it helpful to use the result of (a) in (b).]

Question 15

For each of the following sentences, decide whether or not it is a theorem of Q. If it is a theorem of Q, write down a formal proof showing this. If it is not a theorem of Q, justify this. (You may use without proof the fact that all the axioms of Q are true under the interpretations N^* and N^{**} given in the *Logic Handbook*.)

(i) $\exists x(x + (x \cdot x)) = ((x + x) + x)$

(ii) $\forall x\, x' = (x + \mathbf{0}')$

(iii) $\forall x \neg x'' = (x + \mathbf{0}')$ [11]

Question 16

Explain the meaning of the terms used in the statement of Gödel's First Incompleteness Theorem, and explain what this theorem tells us about arithmetic (i.e. the set of all sentences of the formal language true under the standard interpretation). [11]

[END OF QUESTION PAPER]