Question 2

(i) We prove this by induction of the number n of terms in the product.

Let $P(n)$ be the proposition that if a prime p divides a product $a_1 a_2 \ldots a_n$ of n terms then p divides one of the terms.

$P(1)$ is trivially true, providing the basis for the induction.

So suppose $P(k)$ is true and, aiming to deduce the truth of $P(k+1)$, consider any product $a_1 a_2 \ldots a_k a_{k+1}$. If p divides a_{k+1} then it divides one of the terms. Otherwise $\gcd(p, a_{k+1}) = 1$ and, from p divides $(a_1 a_2 \ldots a_k)a_{k+1}$, Euclid's Lemma gives that p divides $a_1 a_2 \ldots a_k$. But now the induction hypothesis tells us that p divides one of the terms of this product, and the induction step is complete. Hence the result is true by Mathematical Induction.

(ii) As $3n+2$ is not divisible by 3, each prime divisor of $3n+2$ is of one of the forms $3k+1$ or $3k+2$. Now, since

$$(3r+1)(3s+1) = 3(3rs+r+s)+1,$$

it follows that if all the prime divisors were of the form $3k+1$ then so too would be $3n+2$, which is not the case. Hence at least one prime divisor is of the form $3k+2$.

(iii) If $3p+1 = n^2$, where n is a positive integer, then taking the 1 to the right-hand side and factorizing,

$$3p = (n-1)(n+1).$$

Now the left-hand side is a product of two primes, so the two terms on the right-hand side must be these two primes, or one of the terms is 1. That is;

$3 = n-1$ and $p = n+1$, which gives $n = 4$ and $p = 5$,
or $3 = n+1$ and $p = n-1$, which gives $n = 2$ and $p = 1$,
or $1 = n-1$ and $3p = n+1$, which again gives $n = 2$ and $p = 1$.

Thus $p = 5$ is the only possibility.