Question 10

(i) $h = \Pr[f, g]$ where

$$f(x) = s(s(x)) = x + 2,$$
$$g(x, y, z) = \mathrm{dif}(s(z), y) = (z + 1) \dot{-} y,$$

so that

$$h(x, 0) = x + 2,$$
$$h(x, s(y)) = (h(x, y) + 1) \dot{-} y.$$

Thus

(a) $h(3, 0) = 5$,

(b) $h(3, 1) = h(3, s(0)) = (5 + 1) \dot{-} 0 = 6$,
$h(3, 2) = h(3, s(1)) = (6 + 1) \dot{-} 1 = 6$.

(ii) (a) We need to define sum first and then use this to define prod.

$$\mathrm{sum} = \Pr[\mathrm{id}, \mathrm{Cn}[s, \mathrm{id}_3^3]];$$
$$\mathrm{prod} = \Pr[z, \mathrm{Cn}[\mathrm{sum}, \mathrm{id}_1^3, \mathrm{id}_3^3]].$$

(b) $$f(x_1, x_2, 0) = x_2 \cdot x_1^0 = x_2$$

and

$$f(x_1, x_2, s(x_3)) = x_2 \cdot x_1^{x_3+1} = (x_2 \cdot x_1^{x_3}) \cdot x_1,$$

so we can define f recursively by

$$f(x_1, x_2, 0) = x_2,$$
$$f(x_1, x_2, s(x_3)) = f(x_1, x_2, x_3) \cdot x_1.$$

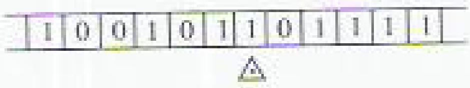
(iii) $f(x_1, x_2, y) = 0$ if and only if both $x_2 \dot{-} (x_1 + y) = 0$ and $(8 \dot{-} x_1) \cdot y = 0$, which happens if and only if both $x_1 + y \geq x_2$ and either $x_1 \geq 8$ or $y = 0$. Thus $\mathrm{Mn}[f](x_1, x_2)$ is defined for those pairs (x_1, x_2) for which $x_1 \geq x_2$ or $x_1 \geq 8$. (There are other equivalent ways of describing the same set of pairs.)

Question 11

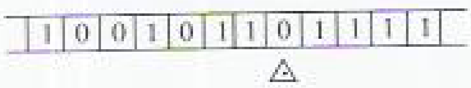
(i)

$$37 = 2^5 + 2^2 + 2^0$$
$$61 = 2^5 + 2^4 + 2^3 + 2^2 + 2^0$$

So the original configuration of the tape is



After the scanning head has moved one square to the right, the configuration is



The new left number is 75 and the new right number is 30.

(ii) (a) We must show that the characteristic function, Eq in the *Logic Handbook*, of the condition is primitive recursive.

Define Eq by

$$\mathrm{Eq}(x, y) = \overline{\mathrm{sg}}(\mathrm{adf}(x, y)),$$

so that

$\mathrm{Eq}(x, y) = 1$ if $\mathrm{adf}(x, y) = 0$, which happens only when $x = y$,

and $\mathrm{Eq}(x, y) = 0$ if $x \neq y$.

Eq is obtained by composing the primitive recursive functions $\overline{\mathrm{sg}}$ and adf, and is thus primitive recursive.