(b) Let c_1 and c_2 be the functions defined as follows:

$$c_1(x,y) = \mathrm{Eq}(\mathrm{prod}(3,x), \mathrm{prod}(4,y));$$
$$c_2(x,y) = \mathrm{Eq}(x, \mathrm{sum}(\mathrm{prod}(2,y),1)).$$

Then

$$\begin{array}{llcll} c_1(x,y) = 1, & \text{if } 3x = 4y, & & c_2(x,y) = 1, & \text{if } x = 2y+1, \\ c_1(x,y) = 0, & \text{otherwise}, & \text{and} & c_2(x,y) = 0, & \text{otherwise}. \end{array}$$

Thus c_1 and c_2 are the characteristic functions of the conditions '$3x = 4y$' and '$x = 2y + 1$' respectively. As they are obtained by composing the primitive recursive functions Eq, sum and prod, both c_1 and c_2 are primitive recursive.

The conditions '$3x = 4y$' and '$x = 2y + 1$' are mutually exclusive, for if $3x = 4y$ then x must be even, while if in addition $x = 2y + 1$, it must also be the case that x is odd, which is impossible.

Let g_1, g_2 and g_3 be the functions defined by

$$g_1(x,y) = \mathrm{prod}(\exp(x,2), y),$$
$$g_2(x,y) = \mathrm{sum}(x, \mathrm{prod}(2,y)),$$
$$g_3(x,y) = 4.$$

As g_1, g_2 and g_3 are all obtained by composing the primitive recursive functions exp, prod and sum, they are primitive recursive.

The result then follows using the 'otherwise' form of definition by cases in the *Logic Handbook*.

(Strictly speaking, constants like the 4 in the definition of $g_3(x,y)$ should be expressed in terms of x and y, e.g.

$$g_3(x,y) = s(s(s(s(z(\mathrm{id}_1^2(x,y)))))).$$

However, unless stated to the contrary, under exam conditions we would usually allow you simply to write down the constants themselves.)

Question 12

(i) If x is divisible by y, there must be exactly one z with $0 \leq z \leq x$ such that $x = y \cdot z$.

So define d by

$$d(x,y) = \sum_{z=0}^{x} \mathrm{Eq}(x, y \cdot z).$$

Then $d(x,y) = 1$ exactly when x is divisible by y, and $d(x,y) = 0$ otherwise. As $\cdot$ and Eq are primitive recursive functions, so is $(x,y,z) \longmapsto \mathrm{Eq}(x, y \cdot z)$. Then by the technique of general summation (page 3 of the *Logic Handbook*), d is primitive recursive.

(ii) Consider $\sum_{y=0}^{x} d(x,y)$. As $d(x,1) = d(x,x) = 1$, this sum will always be greater than or equal to 2, except when $x = 0$ or 1, when the sum take values less than 2.

The sum equals 2 exactly when x is prime. So define p by

$$p(x) = \mathrm{Eq}\left(\left(\sum_{y=0}^{x} d(x,y)\right), 2\right).$$

As d is primitive recursive, general summation gives that $x \longmapsto \sum_{y=0}^{x} d(x,y)$ is primitive recursive. As Eq is also primitive recursive, p is primitive recursive.