**Question 7**

(i)
$$65 = 3 \times 19 + 8$$
$$19 = 2 \times 8 + 3$$
$$8 = 2 \times 3 + 2$$
$$3 = 1 \times 2 + 1$$
$$2 = 2 \times 1$$

So $65/19 = [3, 2, 2, 1, 2]$.

The convergents are $\frac{3}{1}, \frac{7}{2}, \frac{17}{5}, \frac{24}{7}$ and $\frac{65}{19}$. From the final pair we have

$$24 \times 19 - 7 \times 65 = 1.$$

Multiplying by 3 and rearranging, we have

$$65 \times (-21) + 19 \times 72 = 3,$$

giving $x = -21$, $y = 72$ as one solution of $65x + 19y = 3$. The general solution is therefore

$$x = -21 + 19t, \ y = 72 - 65t, \ t \text{ any integer.}$$

(ii) The integer part of $\frac{\sqrt{12}+9}{3}$ is 4 and so, applying the continued fraction algorithm,

$$x_1 = \frac{\sqrt{12}+9}{3} = 4 + \frac{\sqrt{12}-3}{3}; \qquad a_1 = 4$$
$$x_2 = \frac{3}{\sqrt{12}-3} = \sqrt{12}+3 = 6 + (\sqrt{12}-3); \quad a_2 = 6$$
$$x_3 = \frac{1}{\sqrt{12}-3} = \frac{\sqrt{12}+3}{3} = 2 + \frac{\sqrt{12}-3}{3}; \quad a_3 = 2$$
$$x_4 = \frac{3}{\sqrt{12}-3} = x_1,$$

and we are in a cycle.

$$\text{So } \frac{\sqrt{12}+9}{3} = [4, \overline{(6, 2)}].$$

The first five convergents are $C_1 = \frac{4}{1}$, $C_2 = \frac{25}{6}$, $C_3 = \frac{54}{13}$, $C_4 = \frac{349}{84}$ and $C_5 = \frac{752}{181}$.

The convergent $C_3 = \frac{54}{13}$ is known to be accurate to within $\frac{1}{13 \times 84} < 0.001$ but the error in $C_2 = \frac{25}{6}$ is at least $\frac{1}{2 \times 6 \times 13} > 0.001$. So $C_3$ is the required convergent.