Question 15

(i) This is a theorem of Q.

1	(1)	$\forall x\ (x + \mathbf{0}) = x$	Ass
2	(2)	$\forall x\ (x \cdot \mathbf{0}) = \mathbf{0}$	Ass
	(3)	$((\mathbf{0} + \mathbf{0}) + (\mathbf{0} \cdot \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) + (\mathbf{0} \cdot \mathbf{0}))$	II
1	(4)	$(\mathbf{0} + \mathbf{0}) = \mathbf{0}$	UE, (1)
2	(5)	$(\mathbf{0} \cdot \mathbf{0}) = \mathbf{0}$	UE, (2)
1	(6)	$(\mathbf{0} + (\mathbf{0} \cdot \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) + (\mathbf{0} \cdot \mathbf{0}))$	SR, (3), (4)
1, 2	(7)	$(\mathbf{0} + (\mathbf{0} \cdot \mathbf{0})) = ((\mathbf{0} + \mathbf{0}) + \mathbf{0})$	SR, (5), (6)
1, 2	(8)	$\exists x\ (x + (x \cdot x)) = ((x + x) + x)$	EI, (7)

As assumptions 1 and 2 are axioms of Q, we have

$$\vdash_Q \exists x\ (x + (x \cdot x)) = ((x + x) + x).$$

(ii) This is a theorem of Q.

1	(1)	$\forall x\ (x + \mathbf{0}) = x$	Ass
2	(2)	$\forall x\ \forall y\ (x + y') = (x + y)'$	Ass
	(3)	$(x + \mathbf{0}') = (x + \mathbf{0}')$	II
2	(4)	$\forall y\ (x + y') = (x + y)'$	UE, (2)
2	(5)	$(x + \mathbf{0}') = (x + \mathbf{0})'$	UE, (4)
1	(6)	$(x + \mathbf{0}) = x$	UE, (1)
1, 2	(7)	$(x + \mathbf{0}') = x'$	SR, (5), (6)
1, 2	(8)	$x' = (x + \mathbf{0}')$	SR, (3), (7)
1, 2	(9)	$\forall x\ x' = (x + \mathbf{0}')$	UE, (8)

As assumptions 1 and 2 are axioms of Q, we have $\vdash_Q \forall x\ x' = (x + \mathbf{0}')$.

(iii) This is not a theorem of Q.

In this interpretation N^* in the *Logic Handbook*, taking x as ω, we have

$$\omega'' = (\omega + 0') = \omega,$$

so that $-x'' = (x + \mathbf{0}')$ is not true for all x in N^*. Thus $\forall x - x'' = (x + \mathbf{0}')$ is false in N^*.

As the axioms of Q are true in N^*, it follows by the Correctness Theorem that $\forall x - x'' = (x + \mathbf{0}')$ is not a theorem of Q.

Question 16

An *extension* of Q is a theory whose axioms include those of Q. Calling it *consistent* means that in this extension one cannot prove both a sentence ϕ and its negation $-\phi$, for any ϕ. Calling it *complete* means that for all sentences σ either σ or $-\sigma$ is provable in the extension. Calling it *axiomatizable* means that the set of gödel numbers of the axioms (or theorems) of the extension form a recursively enumerable set, so essentially there is an algorithm for generating all the theorems of the theory.

Thus Gödel's First Theorem tells us that arithmetic (which is *consistent*, as all of its sentences are true under the standard interpretation, and which is *complete*, as for all σ in our language σ or $-\sigma$ must be true in this interpretation) cannot be axiomatizable. So there is no set of axioms that we can generate algorithmically from which we can prove precisely all the true sentences of arithmetic.