M381 Solutions to the Specimen Examination Paper

Question 1

(i) Let $P(n)$ be the proposition that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2.$$

$P(1)$ claims $1^3 = \frac{1}{4}1^2(2)^2$ which is seen to be true, so we have a basis for the induction.

Suppose that $P(k)$ is true; that is,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2.$$

Then

$$\begin{aligned} 1^3 + 2^3 + 3^3 &+ \cdots + k^3 + (k+1)^3 \\ &= \frac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ by the induction hypothesis,} \\ &= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\ &= \frac{1}{4}(k+1)^2(k+2)^2, \end{aligned}$$

which shows the truth of $P(k+1)$, and completes the induction step. The result is therefore true for all integers $n \geq 1$ by Mathematical Induction.

(ii)
$$\begin{aligned} 532 &= 1 \times 323 + 209 \\ 323 &= 1 \times 209 + 114 \\ 209 &= 1 \times 114 + 95 \\ 114 &= 1 \times 95 + 19 \\ 95 &= 5 \times 19 + 0 \end{aligned}$$

So $\gcd(532, 323) = 19$, being the last non-zero remainder in the Euclidean Algorithm.

Reversing the steps,

$$\begin{aligned} 19 &= 114 - 95 \\ &= 114 - (209 - 114) = -209 + 2 \times 114 \\ &= -209 + 2(323 - 209) = 2 \times 323 - 3 \times 209 \\ &= 2 \times 323 - 3(532 - 323) = -3 \times 532 + 5 \times 323. \end{aligned}$$

So $x = 5$ and $y = 3$.

(iii) Let $\gcd(2n^2 + 2n + 1, 2n + 1) = d$. Then d divides $2n^2 + 2n + 1$ and d divides $2n + 1$, and so d divides $(2n^2 + 2n + 1) - n(2n + 1) = n + 1$. But then d divides both $n + 1$ and $2n + 1$, and so d divides $2(n + 1) - (2n + 1) = 1$. Hence $d = 1$.

SUP 32047 4