Question 13

(i) Let ϕ be the subformula $x = y$,

ψ the subformula $\forall x\, x = y$

and θ the subformula $\exists x(x = y \vee \forall x\, x = y)$.

Then the formula becomes

$$(((\phi \vee \psi) \longrightarrow \theta) \longrightarrow (-\theta \longrightarrow -\phi))$$

which has truth table

$(((\phi$	$\vee$	$\psi)$	$\longrightarrow$	$\theta)$	$\longrightarrow$	$(-$	θ	$\longrightarrow$	$-$	$\phi))$
1	1	1	1	1	1	0	1	1	0	1
1	1	1	0	0	1	1	0	0	0	1
1	1	0	1	1	1	0	1	1	0	1
1	1	0	0	0	1	1	0	0	0	1
0	1	1	1	1	1	0	1	1	1	0
0	1	1	0	0	1	1	0	1	1	0
0	0	0	1	1	1	0	1	1	1	0
0	0	0	1	0	1	1	0	1	1	0
					†					

This table shows that the formula is true under all interpretations.

(ii) (a) 1 on line (1); 2 on line (2); 3 on line (3);
2 on line (4); 5 on line (5); 2 on line (6);
2,3,5 on line (7); 2,3 on line (8);
2,3 on line (9); 1,2 on line (10).

(b) $((((\phi \longrightarrow \psi) \,\&\, \chi) \,\&\, -\psi) \longrightarrow -(-\chi \vee \phi))$

(c) (A) NO,

(B) YES.

(iii) (a) Take both ϕ and ψ to be $x = x$. (Any ϕ and ψ for which $(\phi \longrightarrow \psi)$ is a tautology will do.)

(b) Take ϕ to be $\forall x\, x = x$ and ψ to be $x = x$.

Question 14

(i) (a) YES;

(b) NO;

(c) YES;

(ii) (a)

1	(1)	$-\exists x\phi$	Ass
2	(2)	ϕ	Ass
2	(3)	$\exists x\phi$	EI, (2)
1, 2	(4)	$(\psi \,\&\, -\psi)$	Taut, (1), (3)

(b) Add the following lines to the proof in (a).

1	(5)	$(\phi \rightarrow (\psi \,\&\, -\psi))$	CP, (4)
1	(6)	$-\phi$	Taut, (5)
1	(7)	$\forall x - \phi$	UI, (6)

(c)

1	(1)	$\forall x - \phi$	Ass
2	(2)	$\exists x\phi$	Ass
3	(3)	ϕ	Ass
1	(4)	$-\phi$	UE, (1)
1, 3	(5)	$(\psi \,\&\, -\psi)$	Taut, (3), (4)
1, 2	(6)	$(\psi \,\&\, -\psi)$	EH, (5)
1	(7)	$(\exists x\phi \rightarrow (\psi \,\&\, -\psi))$	CP, (6)
1	(8)	$-\exists x\phi$	Taut, (7)

(Note that the use of EH would have been invalid, had we used $(\phi \,\&\, -\phi)$ instead of $(\psi \,\&\, -\psi)$ throughout.)