Question 5

A positive integer n has the property that $n + \sigma(n)$ is divisible by 3, where σ denotes the 'sum of the divisors' function. In what follows, p is a prime number and k is a positive integer. Prove each of the following.

(i) If $n = p$ then $p \equiv 1 \pmod 6$). [3]

(ii) $n \neq p^2$; that is, n cannot be equal to the square of a prime. [3]

(iii) $n \neq 2^k$; that is, n cannot be equal to a power of 2. [2]

(iv) If $n = 2p^k$, where p is odd, then $p = 3$. [3]

Question 6

(i) Determine whether or not the quadratic congruence

$$x^2 + 4x + 6 \equiv 0 \pmod{23}$$

has solutions. [4]

(ii) Evaluate the Legendre symbol $(102/67)$. [3]

(iii) Use Gauss' Lemma to show that 2 is a quadratic residue of prime $p \equiv 7 \pmod 8$. [4]

Question 7

(i) Determine a finite continued fraction for $65/19$ and hence find the general solution of the linear Diophantine equation

$$65x + 19y = 3.$$ [5]

(ii) Determine the simple infinite continued fraction of $\dfrac{\sqrt{12}+9}{3}$. Write down the convergents C_1, C_2, C_3, C_4 and C_5 of this ICF and indicate (with brief justification) which is the first convergent accurate to within 0.001 of $\dfrac{\sqrt{12}+9}{3}$. [6]

Question 8

(i) Given that the continued fraction of $\sqrt{15}$ is $[3, (1, 6)]$, find three positive solutions of the Pell's equation

$$x^2 - 15y^2 = 1.$$ [6]

(ii) Find all primitive Pythagorean triples $(60, y, z)$. [5]