Questions 1 to 5 are on Number Theory.
Questions 6 to 9 are on Mathematical Logic.

Number Theory

Question 1 (Unit 7) — 15 marks

(i) Determine the simple continued fraction of $\frac{117}{35}$ and hence obtain the general solution of the linear Diophantine equation

$$117x - 35y = 2.$$ [5]

(ii) Determine the infinite continued fraction of

$$\frac{3+\sqrt{14}}{5}.$$ [5]

(iii) Evaluate the convergents $C_1, C_2, C_3, \ldots, C_7$ of the infinite continued fraction

$$\alpha = [0, 1, 2, 3, 4, 5, \ldots, n, n+1, \ldots]$$

and, without using a calculator, determine the least value of i for which $|\alpha - C_i| < 0.00001$, justifying your answer. [5]

Question 2 (Unit 7) — 10 marks

Let $\alpha = [a_1, a_2, a_3, a_4, \ldots] = [a_1, a_2, x]$, where $x = [a_3, a_4, a_5, \ldots]$.

(i) Show that if a_2 is even then $2\alpha = \left[2a_1, \frac{1}{2}a_2, 2x\right]$. [2]

(ii) Show that if a_2 is a multiple of 3 then $3\alpha = \left[3a_1, \frac{1}{3}a_2, 3x\right]$. [2]

(iii) Given that $\sqrt{10} = [3, \langle 6 \rangle]$, use the formula of part (i) to determine the ICF of $2\sqrt{10}$. (Your working should show how the formula is being used.) Verify your answer by calculating the fourth convergent C_4 of this ICF and expressing it as a decimal fraction, so that you can compare it to the approximate value 6.324555 of $2\sqrt{10}$. [2]

(iv) Write down the ICFs of $3\sqrt{10}$ and $6\sqrt{10}$. [2]

(v) Show that if $a_2 = 1$ and $a_3 > 2$, then $2\alpha = \left[2a_1 + 1, 1, \frac{1}{2}(x-1)\right]$. [2]

Question 3 (Unit 8) — 9 marks

(i) Given that the continued fraction of $\sqrt{23}$ is $[4, \langle 1, 3, 1, 8 \rangle]$, find two positive solutions of the Diophantine equation $x^2 - 23y^2 = 1$. [6]

(ii) From the solutions in part (i), deduce two positive solutions of the Diophantine equation $x^2 + 6xy - 14y^2 = 1$. [3]

Question 4 (Unit 8) — 8 marks

Determine all Pythagorean triples (x, y, z) in which one of the sides is 42. [8]