Question 5 (Unit 8) — 8 marks

The objective of this question is to use the method of infinite descent to show that $\sqrt{7}$ is irrational.

Suppose that $\sqrt{7} = \dfrac{m}{n}$, where m and n are positive integers. Show that

$$\sqrt{7} = \frac{7n - 2m}{m - 2n}$$

and explain how the method of infinite descent can be used to deduce from this that $\sqrt{7}$ is irrational. [8]

Mathematical Logic

Question 6 (Unit 6) — 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $(\exists x(x + t) = y \vee \forall y(y + t) = x)$

(b) $\forall t \exists y(\exists x\ x = y \,\&\, t' = y)$

(c) $\exists y \forall t(\forall x\ x' = t \rightarrow y = x)$ [$2\frac{1}{2}$]

(ii) For each of the formulas in (a), (b) and (c) above, write down the result of substituting the term $(t \cdot x)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x? [$2\frac{1}{2}$]

Question 7 (Unit 6) — 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x\ \psi \vdash \forall x(\phi \rightarrow \psi)$, where ϕ, ψ are any formulas. [4]

(ii) $\exists x(\phi \rightarrow \psi) \vdash (\phi \rightarrow \exists x\ \psi)$, where the variable x does not occur freely in ϕ, indicating at which step(s) of your proof this condition on ϕ is required. [6]

(iii) $\forall x(\phi \,\&\, -\theta) \vdash \exists x((\theta \longleftrightarrow -\psi) \rightarrow -\forall x(\psi \rightarrow -\phi))$ [6]

(iv) $x = y \vdash ((x \cdot z) + (y \cdot z)) = ((y \cdot z) + (x \cdot z))$ [4]

Question 8 (Unit 7) — 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{3} + \mathbf{1}) \neq \mathbf{3}$, i.e. $\vdash_Q -(\mathbf{0}''' + \mathbf{0}') = \mathbf{0}'''$ [7]

(ii) $\vdash_Q \exists y(y' \cdot y) = (y + \mathbf{0}')$ [8]

Question 9 (Unit 7) — 10 marks

One of the following sentences is a theorem of Q, and one is not. Give a formal proof of the sentence that is the theorem of Q, and explain why the other sentence is not a theorem of Q. (You may assume that the axioms of Q are true in both the interpretations N^* and N^{**} given on pages 5 and 6 of the *Logic Handbook*.)

(i) $\forall x \forall y((x \cdot y) + \mathbf{0}) = ((x + \mathbf{0}) \cdot y)$

(ii) $\forall x \forall y(\mathbf{0} + (x \cdot y)) = (x \cdot (\mathbf{0} + y))$ [10]