Questions 1 to 5 are on Number Theory.
Questions 6 to 9 are on Mathematical Logic.

Question 1 (Unit 7) – 15 marks

(i) Determine the simple continued fraction of 115/76 and hence obtain the general solution of the linear Diophantine equation

$$115x - 76y = 2.$$ [5]

(ii) Determine the infinite continued fraction of

$$\frac{4+\sqrt{14}}{2}$$ [5]

(iii) Write down the convergents $C_1, C_2, C_3, \ldots, C_7$ of the infinite continued fraction $\alpha = [1, 2, 1, 4, 1, 6, 1, 8, \ldots]$ and, without using a calculator, determine the least value of n for which $|\alpha - C_n| < 0.001$, justifying your answer. [5]

Question 2 (Unit 7) – 10 marks

In this question we investigate the problem: 'Find squares (greater than 9) which remain squares when their final (units) digit is removed'. For example, 49 is such a number because it is a square and so too is 4, which is what remains when the final digit (9) is removed. Likewise $169 (= 13^2)$ becomes $16 (= 4^2)$ when its final digit is removed.

Suppose that on removing the final digit k of the square m^2 we are left with square n^2.

(i) Show that $m^2 = 10n^2 + k$, and deduce that

$$\frac{m}{n} - \sqrt{10} = \frac{k}{n^2\left(\frac{m}{n} + \sqrt{10}\right)}.$$ [2]

(ii) Consider the case when $k = 1$. Explain how, for this case, one of the theorems of *Unit 7* allows us to deduce that $\frac{m}{n}$ is necessarily a convergent of $\sqrt{10}$. [2]

(iii) Find two solutions of the posed problem in which k, the removed final digit, is 1. (You may assume that the continued fraction of $\sqrt{10}$ is $[3, \langle 6 \rangle]$.) [4]

(iv) Multiplying through the equation $m^2 = 10n^2 + 1$ by d^2 gives

$$(md)^2 = 10(nd)^2 + d^2.$$

By choosing appropriate values of d find, from *each* of your solutions in part (iii), two further solutions to the posed problem (thus giving four new solutions). [2]

Question 3 (Unit 8) – 10 marks

(i) Find three positive solutions of the Diophantine equation

$$x^2 - 6y^2 = 1.$$

(You may assume that the continued fraction of $\sqrt{6}$ is $[2, \langle 2, 4 \rangle]$.) [6]

(ii) From the solutions in part (i) find three positive solutions of the Diophantine equation

$$x^2 - 2xy - 5y^2 = 1.$$ [4]