Question 4 (Unit 8) - 7 marks

One of the numbers 520 and 540 can be written as a sum of two squares and the other cannot. Explain why one cannot be so written and write the other as a sum of two positive squares in two different ways, showing your working. [7]

Question 5 (Unit 8) - 8 marks

Suppose that $\sqrt{10} = \dfrac{m}{n}$, where m and n are positive integers.

(i) Show that $\sqrt{10} = \dfrac{(10n - 3m)}{(m - 3n)}$. [2]

(ii) Show how the method of infinite descent can be used to deduce from part (i) that $\sqrt{10}$ is irrational. [6]

Question 6 (Unit 6) - 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $\forall y\ (\exists t\ (y + x) = t \rightarrow x' = (t \cdot z))$

(b) $\forall t\ \exists x\ (\forall y\ x = y \rightarrow \exists z\ z' = (t \cdot x))$

(c) $(\forall x\ \exists t\ x' = t' \vee \exists y\ (x + y) = t)$ [$2\frac{1}{2}$]

(ii) For each of the formulas in (a), (b) and (c) above, write down the result of substituting the term $(x \cdot t)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x? [$2\frac{1}{2}$]

Question 7 (Unit 6) - 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x(\phi \vee \theta) \vdash \forall x(\theta \vee \phi)$, where θ, ϕ are any formulas. [4]

(ii) $\forall x\phi \vdash (\exists x(-\phi \vee \psi) \rightarrow \psi)$, where the variable x does not occur free in ψ, indicating at which step(s) of your proof this condition on ψ is required. [6]

(iii) $\exists x(-\phi \rightarrow \theta) \vdash (\forall x(\theta \vee (-\phi \,\&\, \psi)) \rightarrow -\forall x(\psi \,\&\, -\theta))$ [6]

(iv) $x = z \vdash (x \cdot (y + z)) = (z \cdot (y + x))$ [4]

Question 8 (Unit 7) - 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{0} + \mathbf{2}) \neq \mathbf{3}$,
that is, $\vdash_Q -(\mathbf{0} + \mathbf{0}'') = \mathbf{0}'''$ [7]

(ii) $\vdash_Q \exists y((y + y) \cdot y) = y'$ [8]

Question 9 ((Unit 7) - 10 marks)

One of the following sentences is a theorem of Q, and one is not. Give a formal proof of the sentence that is a theorem of Q, and explain why the other sentence is not a theorem of Q. (You may assume that the axioms of Q are true in both the interpretations N^* and N^{**} given on pages 5 and 6 of the *Logic Handbook*.)

(i) $\forall x \forall y((x \cdot y) \cdot \mathbf{0}) = (x \cdot (y \cdot \mathbf{0}))$

(ii) $\forall x \forall y((x \cdot \mathbf{0}) \cdot y) = (x \cdot (\mathbf{0} \cdot y))$ [10]