**Question 2** (Unit 7) – 10 marks

Two infinite continued fractions $\alpha = [a_0; a_1, a_2, \ldots]$ and $\beta = [b_0; b_1, b_2, \ldots]$ are said to *eventually agree* if there exist positive integers $m$ and $n$ such that $a_{m+k} = b_{n+k}$ for all integers $k \geq 1$. In other words, if $\alpha$ and $\beta$ eventually agree we may write

$$\alpha = [a_0; a_1, a_2, \ldots, a_m, c_1, c_2, c_3, \ldots]$$

and

$$\beta = [b_0; b_1, b_2, \ldots, b_n, c_1, c_2, c_3, \ldots].$$

(i) Consider the continued fractions (which eventually agree):

$$\alpha = [1; 2, 1, c_1, c_2, c_3, \ldots] \quad \text{and} \quad \beta = [3; 2, c_1, c_2, c_3, \ldots].$$

By putting $x = [c_1, c_2, c_3, \ldots]$ show that

$$\alpha = \frac{4x+3}{3x+2},$$

and obtain a similar expression for $\beta$. By eliminating $x$ from this pair of equations prove that

$$\alpha = \frac{2\beta - 9}{\beta - 5}.$$ [4]

(ii) Prove, generally, that if $\alpha$ and $\beta$ are continued fractions which eventually agree then there exist integers $A$, $B$, $C$ and $D$, where $AD - BC = \pm 1$, such that

$$\alpha = \frac{A\beta + B}{C\beta + D}.$$

[*Hints*: Let $\alpha$ and $\beta$ be as in the preamble, let the convergents of

$$[a_0; a_1, a_2, \ldots, a_m] \quad \text{be} \quad p_0/q_0, p_1/q_1, \ldots, p_m/q_m,$$

and let the convergents of

$$[b_0; b_1, b_2, \ldots, b_n] \quad \text{be} \quad r_0/s_0, r_1/s_1, \ldots, r_n/s_n.$$

Now follow the lines of the particular case in part (i). To obtain the condition $AD - BC = \pm 1$ you will need to use Burton's Theorem 13-7 (page 307).] [6]

**Question 3** (Unit 7 and 8) – 12 marks

(i) Show that the continued fraction of $\sqrt{14}$ is $[3; \overline{1, 2, 1, 6}]$. [6]

(ii) Use the continued fraction in part (i) to find two positive solutions of the Diophantine equation $x^2 - 14y^2 = 1$. [4]

(iii) From the solutions in part (ii) deduce two positive solutions of the Diophantine equation $x^2 + 4xy - 10y^2 = 1$. [2]

**Question 4** (Unit 8) – 6 marks

One of the numbers 585 and 700 can be written as a sum of two squares and the other cannot. Explain why one cannot be so written and write the other as a sum of two positive squares in two different ways. (Your working, as in Burton's Example 12-1 (page 268), should be shown.) [6]

**Question 5** (Unit 8) – 8 marks

Suppose that $\sqrt{5} = \dfrac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Show that

$$\sqrt{5} = \frac{5n - 2m}{m - 2n}.$$

Explain how the method of infinite descent can be used to deduce from this that $\sqrt{5}$ is irrational. [8]