Question 6 (Unit 6) – 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $(\exists x(x + t) = y \vee \forall y(y + t) = x)$

B B F F B B F F

(b) $\forall t \exists y(\exists x\ x = y \,\&\, t' = y)$

B B F F B B B

(c) $\exists y \forall t(\forall x\ x' = t \rightarrow y = x)$ [$2\frac{1}{2}$]

B B B B B B F

(ii) For each of the formulas in (a), (b) and (c) above, write down the result of substituting the term $(t \cdot x)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x? [$2\frac{1}{2}$]

Question 7 (Unit 6) – 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x - \theta \vdash \forall x(\theta \rightarrow \phi)$, where θ, ϕ are any formulas. [4]

(ii) $\exists x(\psi \rightarrow \phi) \vdash (\forall x \psi \rightarrow \phi)$, where the variable x does not occur free in ϕ, indicating at which step(s) of your proof this condition on ϕ is required. [6]

(iii) $\forall x(\phi \,\&\, -\psi) \vdash (\exists x(\theta \longleftrightarrow \psi) \rightarrow -\forall x(\theta \vee -\phi))$ [6]

(iv) $x = z \vdash (x \cdot (y \cdot z)) = (z \cdot (y \cdot x))$ [4]

Question 8 (Unit 7) – 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{4} + \mathbf{1}) \neq \mathbf{3}$,
that is, $\vdash_Q -(0'''' + 0') = 0'''$ [7]

(ii) $\vdash_Q \exists y(y + y) = (y \cdot y)'$ [8]

Question 9 (Unit 7) – 10 marks

One of the following sentences is a theorem of Q, and one is not. Give a formal proof of the sentence that is a theorem of Q, and explain why the other sentence is not a theorem of Q. (You may assume that the axioms of Q are true in both the interpretations N^* and N^{**} given on pages 5 and 6 of the *Logic Handbook*.)

(i) $\forall x\ \forall y\ (x \cdot (y + \mathbf{0})) = ((x \cdot y) + (x \cdot \mathbf{0}))$

(ii) $\forall x\ \forall y\ ((y + \mathbf{0}) \cdot x) = ((y \cdot x) + (\mathbf{0} \cdot x))$ [10]
