Question 3 (Units 7 and 8) — 12 marks

(i) Show that the continued fraction of $\sqrt{15}$ is $[3, \langle 1, 6 \rangle]$. [6]

(ii) Use the continued fraction in part (i) to find three positive solutions of the Diophantine equation $x^2 - 15y^2 = 1$. [4]

(iii) From your solutions in part (ii) deduce three positive solutions of the Diophantine equation $x^2 + 4xy - 11y^2 = 1$. [2]

Question 4 (Unit 8) — 6 marks

Find all primitive Pythagorean triples $(84, y, z)$, in which the even member is 84. [6]

Question 5 (Unit 8) — 7 marks

Use the method of infinite descent to prove that the Diophantine equation

$$4x^3 - y^3 = 2z^3$$ [7]

has no solutions in the positive integers.

Mathematical Logic

Question 6 (Unit 6) — 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $(\exists t(x + t) = y \vee \forall x(y + t) = x)$

(b) $\forall y \exists t(\exists x\ x = y' \ \&\ t' = y)$

(c) $\exists t \forall y(\exists x\ x' = t \rightarrow y = x)$ [$2\frac{1}{2}$]

(ii) For each of the formulas in (a), (b) and (c) above, write down the result of substituting the term $(y + x)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x? [$2\frac{1}{2}$]

Question 7 (Unit 6) — 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x(\phi \ \&\ \psi) \vdash \forall x(\phi \vee \psi)$, where ϕ, ψ are any formulas. [4]

(ii) $\exists x - \psi \vdash (\forall x(\phi \vee \psi) \rightarrow \phi)$, where the variable x does not occur free in ϕ, indicating at which step(s) of your proof this condition on ϕ is required. [6]

(iii) $\forall x(\phi \vee -\theta) \vdash (\exists x(\psi \longleftrightarrow \theta) \rightarrow -\forall x(\psi \ \&\ -\phi))$ [6]

(iv) $x = z \vdash ((z \cdot y) + x) = ((x \cdot y) + z)$ [4]

Question 8 (Unit 7) — 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{0} + \mathbf{2}) \neq \mathbf{4}$,
that is, $\vdash_Q -(\mathbf{0} + \mathbf{0}'') = \mathbf{0}''''$ [7]

(ii) $\vdash_Q \exists y((y \cdot y) + y) = y'$ [8]