Questions 1 to 5 are on Number Theory.
Questions 6 to 9 are on Mathematical Logic.

Question 1 (Unit 7) — 15 marks

(i) Determine the simple continued fraction of $121/84$ and hence obtain the general solution of the linear Diophantine equation

$$121x - 84y = 3.$$ [5]

(ii) (a) Determine the irrational number α which has the periodic continued fraction $[2, 4, \langle 1, 3\rangle]$. [5]

(b) Write down the convergents $C_1, C_2, C_3, \ldots, C_7$ of α and, without using a calculator, find the least value of i for which $|\alpha - C_i| < 0.0005$, justifying your answer. [5]

Question 2 (Unit 7) — 10 marks

Two infinite continued fractions $\alpha = [a_1, a_2, a_3, \ldots]$ and $\beta = [b_1, b_2, b_3, \ldots]$ are said to *eventually agree* if there exist positive integers m and n such that $a_{m+k} = b_{n+k}$ for all integers $k \geq 1$. In other words, if α and β eventually agree then

$$\alpha = [a_1, a_2, \ldots, a_m, c_1, c_2, c_3, \ldots]$$

and

$$\beta = [b_1, b_2, \ldots, b_n, c_1, c_2, c_3, \ldots]$$

for some positive integers $c_1, c_2, c_3, \ldots$.

(i) Consider the continued fractions (which eventually agree)

$$\alpha = [1, 3, c_1, c_2, c_3, \ldots] \text{ and } \beta = [2, 1, 4, c_1, c_2, c_3, \ldots].$$

By putting $x = [c_1, c_2, c_3, \ldots]$ show that

$$\alpha = \frac{4x+1}{3x+1},$$

and obtain a similar expression for β. By eliminating x from this pair of equations prove that

$$\alpha = \frac{\beta - 2}{2\beta - 5}.$$ [5]

(ii) Prove generally that if α and β are continued fractions which eventually agree then there exist integers A, B, C and D, where $AD - BC = \pm 1$, such that

$$\alpha = \frac{A\beta + B}{C\beta + D}.$$

[*Hints*: Let α and β be as in the preamble and let the convergents of

$$[a_1, a_2, \ldots, a_m] \text{ be } p_1/q_1,\ p_2/q_2, \ldots, p_m/q_m,$$

and let the convergents of

$$[b_1, b_2, \ldots, b_n] \text{ be } r_1/s_1,\ r_2/s_2, \ldots, r_n/s_n.$$

Now follow the lines of the particular case in part (i). To obtain the condition $AD - BC = \pm 1$ you will need to use Theorem 1.3 of *Unit 7*.] [5]