Question 4 (Unit 6) — 10 marks

(i) Determine whether the quadratic congruence

$$11x^2 - 15x + 5 \equiv 0 \pmod{23}$$

has solutions

(a) by applying Euler's Criterion;

(b) by using Gauss' Lemma. [4]

(ii) Using the *LQR* in conjunction with Theorems 2.1 and 3.2 of *Unit 6*, determine the values of the Legendre symbols

$$(23/41), (76/103) \text{ and } (547/571).$$

[Note that each of 103, 547 and 571 is prime.] [6]

Question 5 (Unit 6) — 14 marks

(i) Let p be an odd prime. Show that if $(-4/p) = 1$ then p is congruent modulo 8 to either 1 or 5. [3]

(ii) Use the result of part (i) in proving that there are infinitely many primes of the form $8k + 5$, as follows. Suppose that $p_1, p_2, p_3, \ldots, p_n$ are the only primes of the form $8k + 5$. Show that the number $(p_1 p_2 p_3 \ldots p_n)^2 + 4$ must be divisible by some other prime of the form $8k + 5$. [11]

Mathematical Logic

In your answers to Questions 6 and 7 you are advised to show functions to be primitive recursive by the use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units or mentioned in the Logic Handbook.

Question 6 (Unit 4) — 10 marks

Show that the function f of two arguments defined by

$$f(x, y) = \begin{cases} x^2 y, & \text{if } x \cdot y = 120 \text{ and } x \geq 3, \\ x + 3y, & \text{if } x + 2y > 200, \\ 2, & \text{otherwise,} \end{cases}$$

is primitive recursive. [10]

Question 7 (Unit 4) — 10 marks

(i) Show that the function c defined by

$$c(x, y) = \begin{cases} 1, & \text{if } y^4 \leq x, \\ 0, & \text{otherwise,} \end{cases}$$

is primitive recursive. [3]

(ii) Show that the function h defined for natural numbers by

$$h(x) = [x^{1/4}], \quad \text{i.e. the greatest integer less than or equal to the fourth root of } x,$$

is primitive recursive. [7]

[For example, $h(0) = 0$, $h(1) = h(2) = \ldots = h(15) = 1$, $h(16) = 2$.]

[*Hint*: How is the number of ys less than or equal to x for which $y^4 \leq x$ related to the value of $h(x)$?]