Questions 1 to 5 are on Number Theory.
Questions 6 to 10 are on Mathematical Logic.

Number Theory

Question 1 (Unit 5) – 12 marks

(i) Determine all positive integers $n < 500$ which have $\tau(n) = 18$. [4]

(ii) An integer n with $\sigma(n) > 2n$ is said to be an *abundant* number.

(a) Is 380 abundant?

(b) Determine all primes p for which $8p$ is abundant.

(c) Determine all primes p for which $10p$ is abundant.

(d) For which non-negative integers a, if any, is $2^a.11$ abundant? Briefly explain your answer. [8]

Question 2 (Unit 5) – 8 marks

For each positive integer n, let X_n be the set of n^2 ordered pairs of integers

$$X_n = \{(a,b) : 1 \le a \le n, 1 \le b \le n\}.$$

A function F is defined for all positive integers by

$$F(n) = \text{the number of ordered pairs } (a,b) \in X_n \text{ for which } \gcd(a,b) \text{ is relatively prime to } n.$$

For example, $F(4) = 12$ since, of the sixteen elements of X_4

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & \underline{(2,2)} & (2,3) & \underline{(2,4)} \\ (3,1) & \underline{(3,2)} & (3,3) & \underline{(3,4)} \\ (4,1) & \underline{(4,2)} & (4,3) & \underline{(4,4)} \end{array}$$

all but the four underlined pairs have greatest common divisor relatively prime to 12.

(i) Compute $F(2)$, $F(3)$, $F(5)$, $F(6)$ and $F(8)$. [1]

(ii) Determine $F(p)$ for any prime p. [2]

(iii) Determine $F(p^r)$, for any prime p and integer $r > 1$. [3]

(iv) Given that F is a multiplicative function, determine $F(675)$ and complete the following formula by supplying the entry missing from the bracket:

$$F(n) = n^2 \prod_{p|n} (\qquad). \qquad [2]$$

Question 3 (Unit 5) – 6 marks

Suppose that r is a primitive root of the odd prime p. Prove that

$$r^{(p-1)/2} \equiv -1 \pmod{p}.$$

Hence prove that $-r$ is also a primitive root of p if, and only if, $p \equiv 1 \pmod 4$. [*Hint*: Identify which power of r is congruent modulo p to $-r$ and then apply Theorem 5.2 or its Corollary.] [6]