Question 4 (Unit 6) – 11 marks

In Theorem 3.5 of *Unit 6* we proved that there are infinitely many primes of the form $8k+7$. In this question we ask you to prove that same result, but by a slightly different method.

Suppose that there are only finitely many primes of the form $8k+7$, say $p_1, p_2, p_3, \ldots, p_n$. Consider the number

$$N = (p_1 p_2 p_3 \ldots p_n)^2 - 2.$$

(i) Show that $N \equiv 7 \pmod 8$. [1]

(ii) Show that if p is a prime divisor of N then $(2/p) = 1$. [2]

(iii) Deduce that one of the primes $p_1, p_2, p_3, \ldots, p_n$ must be a divisor of N and thereby reach a contradiction. [8]

Mathematical Logic

In your answers to Questions 5 and 6 you are advised to show functions to be primitive recursive by use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units, or mentioned in the Handbook.

Question 5 (Unit 4) – 10 marks

Show that the function f of two arguments defined by

$$f(x,y) = \begin{cases} x + y^2, & \text{if } x \cdot y = 360 \text{ and } x \geq 4, \\ 3^{xy}, & \text{if } x + 3y > 370, \\ 4, & \text{otherwise,} \end{cases}$$

is primitive recursive. [10]

Question 6 (Unit 4) – 10 marks

(i) Show that the function c defined by

$$c(x,y) = \begin{cases} 1, & \text{if } x \text{ is divisible by } y^2, \\ 0, & \text{otherwise,} \end{cases}$$

is primitive recursive. [3]

(ii) Show that the function h defined for positive integers by

$$h(x) = \begin{cases} 1, & \text{if } x \text{ is square-free, i.e. is not divisible by the square} \\ & \text{of any integer except 1,} \\ 0, & \text{otherwise,} \end{cases}$$

is primitive recursive. (Note that, for example, $h(1) = 1$, $h(2) = 1$, $h(4) = 0$, $h(6) = 1$, $h(18) = 0$.) [*Hint*: One method of approach is to consider how the number of ys less than or equal to x for which x is divisible by y^2 is related to the problem of deciding whether x is square-free.] [7]