Questions 1 to 4 are on Number Theory.
Questions 5 to 9 are on Mathematical Logic.

Question 1 (Unit 5) – 13 marks

(i) Determine the smallest positive integer which has exactly 45 distinct positive divisors including itself and 1. [3]

(ii) An integer n with the property $\sigma(n) > 2n$ is said to be *abundant*.

(a) Determine which, if either, of 280 and 290 is abundant?

(b) For which odd primes p is $4p^2$ abundant? Justify your answer. [5]

(iii) For which integers n with $n > 1$ is $\phi(n)$ a divisor of n? Justify your answer. [5]

Question 2 (Unit 5) – 12 marks

A function F is defined for all positive integers by

$$F(n) = \begin{cases} \text{the number of integers in the sequence} \\ 1 \times 2,\ 2 \times 3,\ 3 \times 4,\ \ldots,\ (n-1)n,\ n(n+1) \\ \text{which are relatively prime to } n. \end{cases}$$

So, for example, $F(9) = 3$ because only the three underlined terms of the sequence for $n = 9$

$$\underline{1 \times 2},\ 2 \times 3,\ 3 \times 4,\ \underline{4 \times 5},\ 5 \times 6,\ 6 \times 7,\ \underline{7 \times 8},\ 8 \times 9,\ 9 \times 10.$$

are relatively prime to 9. Note also that $F(1) = 1$.

(i) Write down the values of $F(5), F(7), F(8)$ and $F(15)$. [2]

(ii) Prove that $F(n) = 0$ if, and only if, n is even. [3]

(iii) Determine the value of $F(p)$, where p is any odd prime. [2]

(iv) Determine the value of $F(p^r)$, where p is an odd prime and $r \geq 2$. [3]

(v) Given that F is a multiplicative function, determine that value of $F(675)$ and complete the following formula by supplying the entry missing from the bracket:

$$F(n) = n \prod_{p|n} (\qquad).$$ [2]

Question 3 (Unit 6) – 14 marks

(i) Determine whether the quadratic congruence

$$5x^2 - 15x + 11 \equiv 0 \pmod{23}$$

has solutions:

(a) by applying Euler's Criterion;

(b) by using Gauss' Lemma. [4]

(ii) Using the Law of Quadratic Reciprocity in conjunction with Theorems 2.1 and 3.2 of *Unit 6*, determine the values of the Legendre symbols

$$(19/47), \quad (356/83) \quad \text{and} \quad (853/881).$$

(Note that each of 83, 853 and 881 is prime.) [6]

(iii) Determine all primes $p > 3$ for which $(-6/p) = 1$. [4]