

Question 3 (Unit 6) - 12 marks

- (i) Determine whether the quadratic congruence

$$2x^2 + 7x + 1 \equiv 0 \pmod{17}$$

has solutions:

(a) by applying Euler's Criterion; [3]

(b) by using Gauss' Lemma. [3]

- (ii) By using the Law of Quadratic Reciprocity together with Burton's Theorems 9-2 and 9-6 (pages 192 and 197 respectively), determine the values of the following Legendre symbols:

$$(19/43); (43/683); (683/751).$$

[Note that each of 683 and 751 is prime.] [6]

Question 4 (Unit 6) - 13 marks

- (i) Let $p \neq 5$ be an odd prime. Prove that $(5/p) = 1$ if and only if $p \equiv \pm 1 \pmod{10}$. [4]

- (ii) Prove that there are infinitely many primes of the form $10k - 1$, in the following way. Suppose that there are only finitely many primes of the form $10k - 1$, say p_1, p_2, \dots, p_n . Consider the number

$$N = 4(p_1 p_2 p_3 \dots p_n)^2 - 5.$$

- (a) Show that $N \equiv -1 \pmod{10}$.
(b) Show that if p is a prime divisor of N then $(5/p) = 1$.
(c) Using part (i), deduce that N must have a prime divisor of the form $10k - 1$ which is different from each of $p_1, p_2, p_3, \dots, p_n$. [9]

Mathematical Logic

In your answers to Questions 5 and 6 you are advised to show functions to be primitive recursive by use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units, or mentioned in the Handbook.

Question 5 (Unit 4) - 10 marks

Show that the function f of two arguments defined by

$$f(x, y) = \begin{cases} x^2 y, & \text{if } x + y = 50 \text{ and } x \geq 17, \\ y^x, & \text{if } x + 2y > 86, \\ 7, & \text{otherwise,} \end{cases}$$

is primitive recursive. [10]