Questions 1 to 5 are on Number Theory.
Questions 6 to 9 are on Mathematical Logic.

Number Theory

Question 1 (Unit 7) – 16 marks

(i) Determine the simple continued fraction of $\frac{98}{43}$, and hence obtain the general solution of the linear Diophantine equation

$$98x + 43y = 3.$$ [5]

(ii) Determine the irrational number which has periodic continued fraction

$$[0; 4, \overline{1, 3}].$$ [5]

(iii) Determine the infinite continued fraction of α, where

$$\alpha = \frac{\sqrt{13} - 1}{4}.$$

Write down the convergents $C_0, C_1, C_2, \ldots, C_7$ of α, and find the least value of i for which $|\alpha - C_i| < 0.005$, justifying your answer. [*Hint*: to show that a convergent lacks the wanted accuracy, you may find SAQ 17 helpful.] [6]

Question 2 (Unit 7) – 14 marks

Consider the two continued fractions

$$x = [a; 1, b_1, b_2, b_3, b_4, \ldots] \quad \text{and} \quad y = [a; 1 + b_1, b_2, b_3, b_4, \ldots].$$

(i) Letting $\beta = [b_1; b_2, b_3, b_4, \ldots]$, determine the formulas

$$(\beta + 1)(x - a) = \beta \quad \text{and} \quad (\beta + 1)(y - a) = 1.$$ [4]

(ii) By eliminating β from the formulas in part (i), deduce that

$$x + y = 2a + 1.$$ [2]

(iii) From the formula in part (ii) (noting that $2a + 1$ is an integer) write down the continued fraction of $-x$ and the continued fraction of $-y$. [5]

(iv) Let

$$x_1 = [0; 1, 2, 3, 4, 5, \ldots],$$
$$x_2 = [1; 2, 3, 4, 5, \ldots],$$
$$x_3 = [-2; 3, 1, \overline{2, 4}].$$

Write down the continued fractions of $-x_1, -x_2$ and $-x_3$. [3]

Question 3 (Unit 8) – 6 marks

Find three positive solutions of the Diophantine equation

$$x^2 - 12y^2 = 1.$$

Note that the continued fraction of $\sqrt{12}$ is $[3; \overline{2, 6}]$. [6]

Question 4 (Unit 8) – 6 marks

Find all primitive Pythagorean triples (x, y, z) of positive integers in which one of the members is 84. [6]