Questions 1 to 4 are on Number Theory.
Questions 5 to 9 are on Mathematical Logic.

Number Theory

Question 1 (Unit 5) – 13 marks

(i) Determine all positive integers n less than 120 which have the property that $\tau(n) = 12$. [4]

(ii) An integer n with the property $\sigma(n) > 2n$ is said to be an *abundant* number.

(a) Is 390 abundant? Justify your answer.

(b) For which primes p is $10p$ abundant? Justify your answer.

(c) If n is an abundant number and $\gcd(m,n) = 1$, prove that mn is abundant. [6]

(iii) Prove that if p^2 divides n, where p is prime, then

$$\phi(n) = p\,\phi(n/p).$$ [3]

Question 2 (Unit 5) – 12 marks

Let X_n be the set of n^2 ordered pairs of integers $\{(a,b) : 1 \le a \le n,\ 1 \le b \le n\}$.
A function F is defined for all positive integers by

$$F(n) = \text{ the number of ordered pairs } (a,b) \in X_n \text{ for which } \gcd(a,b,n) = 1.$$

For example, $F(4) = 12$ because, of the sixteen elements of X_4

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & \underline{(2,2)} & (2,3) & \underline{(2,4)} \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & \underline{(4,2)} & (4,3) & \underline{(4,4)}, \end{array}$$

only the four underlined pairs (a,b) fail to satisfy $\gcd(a,b,4) = 1$.

(i) Compute $F(2)$, $F(3)$, $F(5)$, $F(6)$ and $F(8)$. [2]

(ii) Determine $F(p)$, for any prime p. [2]

(iii) Determine $F(p^r)$, for any prime p and $r \ge 2$. [4]

(iv) Given that F is a multiplicative function, determine $F(675)$ and complete the following formula by supplying the entry missing from the bracket:

$$F(n) = n^2 \prod_{p|n} (\qquad).$$ [4]

Question 3 (Unit 6) – 12 marks

(i) Determine whether or not the quadratic congruence

$$11x^2 - 15x + 5 \equiv 0 \pmod{23}$$

has solutions:

(a) by using Euler's Criterion;

(b) by using Gauss' Lemma. [6]

(ii) By using the Law of Quadratic Reciprocity together with Burton's Theorems 9-2 and 9-6, determine the values of the following Legendre symbols:

$$(23/41); \quad (76/103); \quad (547/571).$$

Note that, with the exception of 76, all these numbers are prime. [6]