Question 5 (Unit 8) – 8 marks

Use the method of infinite descent to prove that the Diophantine equation

$$x^3 + 2y^3 = 4z^3$$

has no solutions in which x, y and z are all positive. [*8*]

Mathematical Logic

Question 6 (Unit 6) – 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $\forall y(x = t \,\&\, \exists z \forall t(x + y) = (z \cdot t))$

(b) $\exists y(\forall x\, x' = y \vee \exists t(z \cdot t) = x)$

(c) $\forall y\, \exists x(\forall z(z + y) = y \rightarrow x' = y)$ [$2\frac{1}{2}$]

(ii) For each of the formulas in (a), (b), (c) above, write down the result of substituting the term $(z + x)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x? [$2\frac{1}{2}$]

Question 7 (Unit 6) – 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x - \theta \vdash \forall x(\theta \rightarrow \phi)$, where θ, ϕ are any formulas. [*4*]

(ii) $\exists x - \psi \vdash (\forall x(\phi \vee \psi) \rightarrow \phi)$, where the variable x does not occur free in ϕ, indicating at which step(s) of your proof this condition on ϕ is required. [*6*]

(iii) $\exists x(-\theta \,\&\, \psi) \vdash (\forall x(\psi \rightarrow -\phi) \rightarrow -\forall x(\phi \vee \theta))$ [*6*]

(iv) $x = z \vdash ((x \cdot y) + z) = ((z \cdot y) + x)$ [*4*]

Question 8 (Unit 7) – 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{3} + \mathbf{2}) \neq \mathbf{2}$,

that is, $\vdash_Q -(0''' + 0'') = 0''$ [*7*]

(ii) $\vdash_Q \exists y((y \cdot y) + y) = y'$ [*8*]

Question 9 (Unit 7) – 10 marks

One of the following sentences is a theorem of Q, and one is not. Give a formal proof of the sentence that is the theorem of Q, and explain why the other sentence is not a theorem of Q. (You may assume that the axioms of Q are true in both the interpretations N^* and N^{**} given on pages 5 and 6 of the *Logic Handbook*.)

(i) $\forall x \forall y((x \cdot y) + 0) = ((x + 0) \cdot y)$

(ii) $\forall x \forall y(0 + (x \cdot y)) = (x \cdot (0 + y))$ [*10*]