Question 4 (Unit 6) - 13 marks

(i) Suppose that p divides $8m^2 - 1$, where p is prime and m is an integer.

(a) Show that $(8/p) = 1$.

(b) Deduce that $p \equiv 1 \pmod 8$ or $p \equiv 7 \pmod 8$. [5]

(ii) Use the result of part (i) to prove that there are infinitely many primes of the form $8k+7$, in the following way:

Suppose that $p_1, p_2, \ldots, p_n$ are the only primes of the form $8k+7$, and consider $N = 8(p_1p_2 \ldots p_n)^2 - 1$. Determine that each prime divisor of N is either of the form $8k+1$ or of the form $8k+7$. Show that N must have at least one prime divisor of the form $8k+7$ different from $p_1, p_2, \ldots, p_n$. [8]

Mathematical Logic

In your answers to Questions 5 and 6 you are advised to show functions to be primitive recursive by use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units, or mentioned in the Logic Handbook.

Question 5 (Unit 4) - 10 marks

Show that the function f of two arguments defined by

$$f(x,y) = \begin{cases} x^2 + y, & \text{if } x \cdot y = 240 \text{ and } y > 3, \\ x^{2y}, & \text{if } 2x + y \geq 370, \\ 4, & \text{otherwise}, \end{cases}$$

is primitive recursive. [10]

Question 6 (Unit 4) - 10 marks

(i) Show that the function c defined by

$$c(x,y) = \begin{cases} 1 & \text{if } y^3 \leq x \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive. [3]

(ii) Show that the function h defined for natural numbers by

$$h(x) = [x^{1/3}],$$

i.e. the greatest integer less than or equal to the fourth root of x, is primitive recursive. [7]

Note, for example, $h(0) = 0$, $h(1) = h(2) = \ldots = h(7) = 1$, $h(8) = 2$. [*Hint*: how is the number of ys less than or equal to x for which $y^3 \leq x$ related to the value of $h(x)$?]