Question 4 (Unit 6) – 14 marks

(i) Determine whether or not the quadratic congruence

$$3x^2 - 11x + 9 \equiv 0 \pmod{17}$$

has solutions:

(a) by using Euler's Criterion; [3]

(b) by using Gauss' Lemma. [3]

(ii) Using the *LQR* in conjunction with Theorems 2.1 and 3.2 of *Unit 6*, determine the values of the Legendre symbols

$$(30/71),\ (99/37) \text{ and } (211/251).$$

[Note that each of 37, 71, 211 and 251 is prime.] [8]

Question 5 (Unit 6) – 11 marks

Let $N = 5n^2 - 1$ for some positive integer n, such that N has an odd prime divisor p.

(i) Prove that $(5/p) = 1$. [1]

(ii) Use the result of part (i) to show that p has one of the forms $5k + 1$ or $5k + 4$. [2]

(iii) Deduce from part (ii) that if N is odd, then N must have a prime divisor of the form $5k + 4$. [2]

(iv) Prove that there are infinitely many primes of the form $5k + 4$ by the following method. Suppose to the contrary that there are only finitely many primes of this form, say $p_1, p_2, \ldots, p_r$. Now construct from these primes an appropriate value for n so that the corresponding prime whose existence is deduced in part (iii) is not one of the listed primes. [6]

Mathematical Logic

In your answers to Questions 6 and 7 you are advised to show functions to be primitive recursive by use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units, or mentioned in the Handbook.

Question 6 (Unit 4) – 10 marks

Show that the function f of two arguments defined by

$$f(x,y) = \begin{cases} x^3y, & \text{if } x + y = 60 \text{ and } x > 19, \\ 2^y, & \text{if } x + 2y \geq 105, \\ 7, & \text{otherwise,} \end{cases}$$

is primitive recursive. [10]