TMA 03 M381

1) a) $n+2\sigma(n)=0 \pmod 3$

n prime $\Rightarrow \sigma(n)=n+1$

$n+2\sigma(n)=n+2(n+1)=3n+2=2 \pmod 3 \neq 0 \pmod 3$

$\therefore$ n not prime

b) $n=p^2$, $2\sigma(n)=2\sigma(p^2)=2(1+p+p^2)$

$n+2\sigma(n)=p^2+2(1+p+p^2)$

$=p^2+2+2p+2p^2=2+2p+3p^2 \equiv 2+2p \pmod 3$

$p \equiv 0, 1$ or $2 \pmod 3$

$p \equiv 0 \pmod 3 \Rightarrow 2+2p \equiv 2 \pmod 3 \neq 0 \pmod 3$

$\therefore p \not\equiv 0 \pmod 3$

$p \equiv 1 \pmod 3 \Rightarrow 2+2p \equiv 2+2 \pmod 3 = 1 \pmod 3 \neq 0 \pmod 3$

$\therefore p \not\equiv 1 \pmod 3$

$p \equiv 2 \pmod 3 \Rightarrow 2+2p = 2+2\cdot 2 \pmod 3 \equiv 0 \pmod 3$

$\therefore$ by ~~elimination~~ $p \equiv 2 \pmod 3$

c) $n=2p$

6

$n+2\sigma(n)=2p+2\sigma(2p)=2p+2\sigma(2)\sigma(p)$ unless $p=2$

$=2p+2(2+1)(p+1)=2p+6(p+1)$

$2p+6(p+1)=0 \pmod 3$

$\Rightarrow 2p=0 \pmod 3 \Rightarrow p=3k$

The only prime for which $2p=0 \pmod 3$ is $p=3$ since $k>1 \Rightarrow p$ composite; $p=2$ also works.

2) If $n=p_1^{k_1}p_2^{k_2}p_3^{k_3} \quad p_n^{k_n}$

$\phi(n)=n\prod\left(1-\frac{1}{p_i}\right)$

$=p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots p_n^{k_n}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right)$

$=p_1^{k_1-1}(p_1-1)\,p_2^{k_2-1}(p_2-1)\cdots p_n^{k_n-1}(p_3-1)$