Questions 1 to 5 are on Number Theory.
Questions 6 to 10 are on Mathematical Logic.

Number Theory

Question 1 (Unit 5) — 7 marks

Suppose that an integer n has the property that $n + 2\sigma(n)$ is divisible by 3, (where $\sigma(n)$ denotes the sum of the divisors of n).

(a) Show that n cannot be prime. [1]

(b) Show that if $n = p^2$, where p is prime, then $p \equiv 2 \pmod 3$. [3]

(c) For which primes p is $n = 2p$ possible? Justify your answer. [3]

Question 2 (Unit 5) — 7 marks

Prove that if a positive integer n is divisible by two distinct odd primes then $\phi(n)$ is divisible by 4. Hence determine all positive integers n which have the property that $\phi(n)$ is *not* divisible by 4. (ϕ is Euler's phi-function.) [7]

Question 3 (Unit 5) — 11 marks

A function F is defined for all positive integers by

$F(n)$ = the number of integers a in the range $1 \leq a \leq n$
for which $\gcd(a, n)$ is square-free.

[Recall that an integer is said to be *square-free* if it is not divisible by the square of any prime.]

So, for example, $F(12) = 9$ since $\gcd(4, 12)$, $\gcd(8, 12)$ and $\gcd(12, 12)$ are each divisible by 2^2, while $\gcd(a, 12)$ is square-free for the remaining nine values, $a = 1, 2, 3, 5, 6, 7, 9, 10$ and 11.

(i) Confirm that $F(16) = 12$ by listing the integers a in the range $1 \leq a \leq 16$ for which $\gcd(a, 16)$ is square-free. [1]

(ii) Write down the values of $F(7)$, $F(8)$, $F(9)$ and $F(11)$. [2]

(iii) Determine $F(p)$ for any prime p. [2]

(iv) Determine $F(p^r)$ for any prime p and integer $r > 1$. [3]

(v) Given that F is a multiplicative function, determine $F(500)$ and $F(540)$. [2]

(vi) Complete the following formula by supplying the entry missing from the bracket:

$$F(n) = n \prod_{p^2 | n} (\quad),$$

where the product is taken over all primes p for which p^2 divides n, and takes value 1 if there are no such primes. [1]