Questions 1 to 5 are on Number Theory.
Questions 6 to 10 are on Mathematical Logic.

Number Theory

Question 1 (Unit 3) – 10 marks

(i) Solve the linear congruences

(a) $4x \equiv 5 \pmod{7}$;

(b) $4(x+7) \equiv 10 \pmod{13}$;

(c) $8x \equiv 4 \pmod{11}$. *[6]*

(ii) Determine the least positive integer which satisfies all three congruences in part (i) simultaneously. *[4]*

Question 2 (Unit 3) – 12 marks

The parts of this question lead to a solution of the following.

Problem: Which integers (if any) have a square which end in four equal digits (i.e. end in 0000, or 1111, or 2222 etc)?

(i) Write down which digits cannot occur as the final digit of a square. *[1]*

(ii) Using the facts that $n^2 \equiv 0$ or $1 \pmod{4}$ and that an integer N is congruent modulo 4 to the number made up of the last two digits of N, show that a square cannot end in the pair of digits 11. Which other pairs of equal digits can be eliminated similarly? *[3]*

(iii) Determine all integers r, $0 \leq r < 16$, to which a square may be congruent modulo 16. *[3]*

(iv) Prove that any integer terminating in the four equal digits 4444 is congruent modulo 16 to 12. Can such a number be a square? Explain your answer. *[3]*

(v) Answer the problem given in the preamble. *[2]*

Question 3 (Unit 4) – 12 marks

All parts of this question involve FLT in one or other of its forms.

(i) Find the least positive residue of $22^{63} \pmod{31}$. *[3]*

(ii) Solve $19x \equiv 1 \pmod{31}$. Explain why the least positive residue of $19^{59} \pmod{31}$ is a solution of this congruence. Write down a linear congruence modulo 31 whose solution is congruent to $19^{58} \pmod{31}$ and hence determine the least positive residue of

$$19^{58} \pmod{31}.$$ *[5]*

(iii) Prove that $a^{25} \equiv a \pmod{195}$ for every integer a. *[4]*

Question 4 (Unit 4) – 6 marks

Use Wilson's Theorem and its converse in proving that the integer $6k+1$ is prime if, and only if,

$$(6k-3)! + k \equiv 0 \pmod{6k+1}.$$ *[6]*