Questions 1 to 5 are on Number Theory.
Questions 6 to 8 are on Mathematical Logic.

Number Theory

Question 1 (Unit 1) – 11 marks

Use mathematical induction to prove that the following formula, which gives the sum of the first n heptagonal numbers, holds for all integers $n \geq 1$.

$$1 + 7 + 18 + 34 + \cdots + \tfrac{1}{2}n(5n-3) = \tfrac{1}{6}n(n+1)(5n-2)$$ [*11*]

Question 2 (Unit 1) – 12 marks

(i) For each integer n, its square can be written uniquely as $n^2 = 10k + r$, where $0 \leq r \leq 9$. Determine which values of r can occur and which cannot. [*4*]

(ii) For $n > 4$, prove that 10 divides $n!$. [*4*]

(iii) Determine all integers n for which $1! + 2! + 3! + \cdots + n!$ is a square. [*Hint:* Consider the units digit of $1! + 2! + 3! + \cdots + n!$ when $n > 4$.] [*4*]

Question 3 (Unit 1) – 10 marks

Use the Euclidean Algorithm to determine integers x and y such that

$$\gcd(238, 581) = 238x + 581y.$$

Write down the general solution of this linear Diophantine equation and find the particular solution in which y takes its least positive value. [*10*]

Question 4 (Units 1 and 2) – 20 marks

For each of the following statements decide whether it is true or false. If true prove it; if false give a counterexample. (Throughout a, b, k, m and n are integers, and p is a prime.)

(i) If $\gcd(a,b) = p$ then $\gcd(ap, bp^2) = p^2$.

(ii) If $m = 3a + b$ and $n = 5a + 2b$ then $\gcd(m,n) = \gcd(a,b)$.

(iii) Any number of the form $14k + 3$, for $k \geq 0$, must have a prime divisor of this same form.

(iv) If $p > 3$ then p^2 is of the form $12n + 1$. [*20*]

Question 5 (Unit 2) – 12 marks

Let $n \geq 1$ be any integer and p a prime.

(i) Show that if p divides $n(n+1)$ then either p divides n or p divides $n+1$, but not both. [*2*]

(ii) Suppose that $n(n+1)$ is a square. Deduce from part (i) that, in this case, each of n and $n+1$ must be square. [*5*]

(iii) Deduce that, for $n \geq 1$, $n(n+1)$ is never a square. [*5*]