Questions 1 to 5 are on Number Theory.
Questions 6 to 8 are on Mathematical Logic.

Number Theory

Question 1 (Unit 1) – 12 marks

Use the Euclidean Algorithm to determine integers x and y such that

$$\gcd(203, 522) = 203x + 522y.$$

Write down the general solution of this linear Diophantine equation and find the particular solution in which x takes its least positive value. [12]

Question 2 (Unit 1) – 12 marks

(i) Let $m = n(n+1)$ be the product of two consecutive integers.

(a) If $m = 6k + r$, where k and r are integers with $0 \leq r \leq 5$, prove that either $r = 0$ or $r = 2$. [3]

(b) If $m = 10k + r$, where k and r are integers with $0 \leq r \leq 9$, determine the values of r which are possible. [4]

(ii) Using the result of part (i)(b) show that the final digit of a triangular number cannot be 2, 4, 7 or 9. [5]

Question 3 (Unit 1) – 9 marks

Prove, by mathematical induction, that the following formula holds for all integers $n \geq 1$.

$$1 + 5 + 12 + \cdots + \tfrac{1}{2}n(3n-1) = \tfrac{1}{2}n^2(n+1)$$ [9]

Question 4 (Unit 2) – 20 marks

For each of the following statements, decide whether it is true or false. If it is true, prove it; if it is false, give a counter-example. (Throughout a and b are positive integers, n is a non-negative integer and p is a prime.)

(i) If $\gcd(a, b) = p$, then $\gcd(ab, bp) = p^2$.

(ii) Any number of the form $29n + 2$ must have a prime divisor of this same form.

(iii) If $a^2 + b^2 = p^2$ then $\gcd(a, b) = 1$.

(iv) If p^2 divides $n!$ then $p \leq n/2$. [20]

Question 5 (Unit 2) – 12 marks

Prove that there are infinitely many primes of the form $3k + 2$. (Your answer should follow the lines of Theorem 3.6, **B** page 61, or SAQ 14.) [12]