Questions 1 to 5 are on Number Theory.
Questions 6 to 8 are on Mathematical Logic.

## Number Theory

**Question 1** (Unit 1) – 10 marks

Use the Euclidean Algorithm to determine integers $x$ and $y$ such that

$$\gcd(119, 544) = 119x + 544y.$$

Write down the general solution of this linear Diophantine equation and find the particular solution in which $x$ takes its least positive value. [10]

**Question 2** (Unit 1) – 16 marks

Let $n = 12k + 10$, for some integer $k$. Determine the remainders when $n^2$, $n^3$ and $n^4$ are each divided by 12. Conjecture a general result concerning the remainder on dividing powers of $n$ by 12 and, using mathematical induction, prove your result.

Hence show that each of the numbers in the sequence

$$108, 1008, 10008, 100008, \ldots$$

is divisible by 12. [16]

**Question 3** (Units 1 and 2) – 20 marks

For each of the following statements, decide whether it is true or false. If true, prove it; if false, give a counter-example. (Throughout $a$ and $b$ are positive integers and $p$ is an odd prime.)

(i) If $\gcd(a, b) = p$, then $\gcd(a^2, bp) = p^2$.

(ii) There is no prime $p$ such that $3p + 2$ is a square.

(iii) Any number of the form $14a + 3$ must have a prime divisor of this same form.

(iv) If $a \geq 6$ is composite, then $a$ is a divisor of $(a-1)!$. [20]

**Question 4** (Units 1 and 2) – 10 marks

Consider the sequence of numbers

$$4, 7, 12, 19, 28, 39, 52, \ldots, n^2 + 3, \ldots.$$

In this question you are asked to prove that, for any two consecutive terms of this sequence, either they are relatively prime or, as in the case of the pair 39 and 52, the only prime which divides both is 13.

Suppose that some prime $p$ divides both $n^2 + 3$ and $(n+1)^2 + 3$.

(i) Burton's Theorem 2–2(7) tells us that $p$ divides

$$a(n^2 + 3) + b((n+1)^2 + 3)$$

for any integers $a$ and $b$. By finding appropriate values of $a$ and $b$, prove that $p$ divides $2n + 1$. [3]

(ii) Use the information that $p$ divides $n^2 + 3$ and $p$ divides $2n + 1$ to deduce that $p$ divides $n - 6$. [3]

(iii) Prove that $p$ divides 13. [4]