Questions 1 to 4 are on Number Theory.
Questions 5 to 9 are on Mathematical Logic.

## Number Theory

**Question 1** (Unit 3) – 10 marks

(i) Solve the following linear congruences.

(a) $5x \equiv 1 \pmod{7}$

(b) $7(x-2) \equiv 2 \pmod{11}$

(c) $6x \equiv 2 \pmod{17}$ [6]

(ii) Determine the least positive integer which simultaneously satisifies all three of the congruences in part (i). [4]

**Question 2** (Unit 3) – 13 marks

(i) Prove that $10^k \equiv 4 \pmod{6}$ for all positive integers $k$. [4]

(ii) Prove that if $m \equiv n \pmod{6}$ where $m \geq 0$ and $n \geq 0$ are integers, then $10^m \equiv 10^n \pmod{7}$.

[Hint: Write $m = n + 6k$ and show that $10^6 \equiv 1 \pmod{7}$.] [5]

(iii) Using parts (i) and (ii), determine the remainder on dividing the number

$$10^{10} + 10^{(10^2)} + 10^{(10^3)} + \cdots + 10^{(10^{10})}$$

by 7. [4]

**Question 3** (Unit 4) – 12 marks

(i) Determine the remainder when $7^{39}$ is divided by 37. [3]

(ii) Solve $11x \equiv 1 \pmod{37}$, and explain why the least positive solution of this congruence is equal to the least positive residue of $11^{35} \pmod{37}$. Determine the least positive residue of $11^{34} \pmod{37}$. [5]

(iii) Prove that $a^{37} \equiv a \pmod{285}$ for every integer $a$. [4]