Questions 1 to 5 are on Number Theory.
Questions 6 to 8 are on Mathematical Logic.

Number Theory

Question 1 (Unit 1) – 12 marks

Use the Euclidean Algorithm to determine integers x and y such that

$$\gcd(242, 176) = 242x + 176y.$$

Write down the general solution of this linear Diophantine equation, and find the particular solution in which y takes its least possible value. [*12*]

Question 2 (Unit 1) – 11 marks

(i) Let $m = n(n+1)$ be the product of two consecutive integers.

(a) If $m = 6k + r$, where k and r are integers with $0 \leq r \leq 5$, prove that either $r = 0$ or $r = 2$. [*4*]

(b) If $m = 10k + r$, where k and r are integers with $0 \leq r \leq 9$, determine the values of r which are possible. [*4*]

(ii) Using the result of part (i)(b), show that the final digit of a triangular number cannot be 2, 4, 7 or 9. [*3*]

Question 3 (Unit 1) – 10 marks

Prove, by mathematical induction, that the following formula holds for all integers $n \geq 1$:

$$\frac{3}{1^2 2^2} + \frac{5}{2^2 3^2} + \frac{7}{3^2 4^2} + \cdots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$ [*10*]

Question 4 (Unit 2) – 20 marks

For each of the following statements, decide whether it is true or false. If it is true, then prove it; if it is false, give a counter-example.

(i) If $\gcd(a, b) = p$, then $\gcd(ap, bp^2) = p^2$. (Here a and b are positive integers and p is prime.)

(ii) Any number of the form $10k + 3$ (where $k \geq 0$ is an integer) must have a prime divisor of this same form.

(iii) It is impossible to find three primes p, q and r such that $p > 3$, $q = 4p + 1$ and $r = 4q + 1$.

(iv) No number of the form $n^3 + 1$, where $n > 1$ is an integer, is prime. [*20*]

Question 5 (Unit 2) – 12 marks

Prove that there are infinitely many primes of the form $3k + 2$, where k is an integer. (Your answer should follow the lines of Theorem 3.3, and will require you to prove a result corresponding to Theorem 3.2.) [*12*]