⑤

3) a) $\sum \frac{4^n}{n^2}(z-i)^n$

Use the ratio test.

$\left|\frac{4^{n+1}}{(n+1)^2}(z-i)^{n+1}\right| \div \left|\frac{4^n}{n^2}(z-i)^n\right| < 1$ for convergence

$\left| 4 \times \left(\frac{n}{n+1}\right)^2 (z-i) \right| < 1$

$\left| 4\left(\frac{n}{n+1}\right)^2 \right| |(z-i)| < 1$ ✓

$\therefore |z-i| < \frac{(n+1)^2}{4n^2} \to \frac{1}{4}$ as $n \to \infty$

3A, M $\therefore$ Disc of convergence is $|z-i| < \frac{1}{4}$ ✓

b) $f(z) = z\,\mathrm{Log}\,z$

$f'(z) = \frac{d}{dz}(z\,\mathrm{Log}\,z) = \mathrm{Log}\,z + 1$ ✓

1M $f''(z) = \frac{d}{dz}(\mathrm{Log}\,z) = \frac{1}{z}$ ✓

$f'''(z) = \frac{d}{dz}(z^{-1}) = -z^{-2}$ ✓

2A $f^{iv}(z) = \frac{d}{dz}(-z^{-2}) = 2z^{-3}$ ✓

⋮

Does $f^n(z) = (-1)^n (n-2)!\, z^{-(n-1)}$? ✓

1A $f^{n+1}(z) = \frac{d}{dz}\left((-1)^n (n-2)!\, z^{-(n-1)}\right)$

$= (-1)^n \times -(n-1)(n-2)!\, z^{-(n-1)-1}$

$= (-1)^{n+1}(n-1)!\, z^{-n}$ $\therefore$ true by induction.

$f(z) = \sum_{n=0}^{n=\infty} \frac{f^n(a)}{n!}(z-a)^n$ ✓

$= (z\,\mathrm{Log}\,z)_i + (\mathrm{Log}\,z+1)_i (z+i) + \sum_{n=2}^{\infty} \frac{(-1)^n \cancel{(n-1)!}\,(-i)^{-n+1}(z+i)^n}{n!}$ $(n-2)!$

$= -i\,\mathrm{Log}(-i) + (\mathrm{Log}(-i)+1)(z+i) + \sum_{n=2}^{\infty} \frac{(-1)^n (-i)^{-n+1}(z+i)^n}{n(n-1)}$

$\frac{1}{2}$ A $= -i\left(\frac{-\pi i}{2}\right) + \left(\frac{-\pi i}{2}+1\right)(z+i) + \sum_{n=2}^{\infty} \frac{(-1)^n(-i)^{-n+1}(z+i)^n}{n}$

$= -\frac{\pi}{2}$ ✓ $+ \left(1 - \frac{\pi i}{2}\right)(z+i)$ ✓ $+ \sum_{n=2}^{\infty} \frac{(-1)^n(-i)^{-n+1}(z+i)^n}{n(n-1)}$ ✓