Answer *all* questions.

Question 1 (*Unit B1*) – 25 marks

(a) (i) Evaluate $\int_{\Gamma_1} (\operatorname{Im} z)^3\,dz$, where $\Gamma_1 : \gamma_1(t) = t^2 + it\ (t \in [0,2])$

(ii) Evaluate $\int_{\Gamma_2} (\operatorname{Im} z)^3\,dz$, where $\Gamma_2 : \gamma_2(t) = (4+2i)t\ (t \in [0,1])$

(iii) Use your answers to parts (i) and (ii) to deduce that the function $f(z) = (\operatorname{Im} z)^3$ cannot be the derivative of an entire function. [*12*]

(b) Use the Estimation Theorem to find an upper estimate for the modulus of the integral

$$\int_\Gamma \frac{e^{3z}+z}{z^3+z}\,dz,$$

where Γ is the upper half of the circle $\{z : |z| = 10\}$ traversed anticlockwise from 10 to -10. [*7*]

(c) Use the Fundamental Theorem of Calculus to evaluate the following integrals

(i) $\int_\Gamma e^{-z}\,dz$

(ii) $\int_\Gamma \frac{1}{z^2}\cos\left(\frac{1}{z}\right)dz$,

where Γ is the same contour as in part (b). [*6*]

Question 2 (*Unit B2*) – 25 marks

(a) Evaluate the integral

$$\int_C \frac{\sin z}{(z+4)^2(z-2)}\,dz$$

(i) when $C = \{z : |z| = 1\}$;

(ii) when $C = \{z : |z| = 3\}$;

(iii) when $C = \{z : |z+4| = 1\}$. [*13*]

(b) (i) Use Cauchy's Estimate (*Unit B2*, page 26, Problem 3.3) to give a proof of Liouville's Theorem.

(*Hint*: Show that if f is a bounded entire function, then $f'(\alpha) = 0$, for all $\alpha \in \mathbb{C}$.)

(ii) Prove that there does not exist any entire function f satisfying

$$|f(z)| > |z|, \qquad \text{for } z \in \mathbb{C}.$$

(*Hint*: Apply Liouville's Theorem to the function $g(z) = z/f(z)$.)

(iii) Give an example of an entire function f such that the inequality

$$|f(z)| > |z|, \qquad \text{for } |z| > 0,$$

is true. [*12*]