Question 3 (*Unit B3*) – 25 marks

(a) Determine the disc of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{4^n}{n^2}(z-i)^n.$$ [3]

(b) Use Taylor's Theorem to determine the Taylor series about $-i$ for the function $f(z) = z \operatorname{Log} z$, giving an expression for the general term of the series. Also, state the largest open disc on which the function f is represented by this Taylor series. [7]

(c) Find the Taylor series about 0 for each of the following functions.

(i) $f(z) = e^{-z} \operatorname{Log}(1+2z)$ (up to the term in z^3)

(ii) $f(z) = \cos(\sin 2z)$ (up to the term in z^4) [10]

(d) Let the function f be entire and suppose that

$$f(iy) = e^{-y}, \quad \text{for } y \in \mathbb{R}.$$

Show that $f(z) = e^{iz}$, for all $z \in \mathbb{C}$. [5]

Question 4 (*Unit B4*) – 25 marks

(a) Locate and classify the singularities (giving the order of any poles) of the following function.

$$f(z) = \frac{z}{1-\cos z}$$ [8]

(*Hint:* Use $\cos z = \cos(z - 2k\pi)$, for $k \in \mathbb{Z}$.)

(b) Let $f(z) = \left(1 - \dfrac{1}{z^2}\right)\cosh\left(\dfrac{1}{z}\right)$.

(i) Find the Laurent series for the function f about 0, giving the general term of the series.

(ii) Write down a punctured open disc D, containing the circle $C = \{z : |z| = 1\}$, on which f is represented by this series.

(iii) State the nature of the singularity of f at 0.

(iv) Evaluate

$$\int_C \left(1 - \frac{1}{z^2}\right)\cosh\left(\frac{1}{z}\right) dz,$$

where $C = \{z : |z| = 1\}$. [9]

(c) Find the Laurent series about 0 for the function

$$f(z) = \frac{z}{(3z-1)(z+3)}$$

on the set $\{z : |z| > 3\}$, giving the general term of the series. [8]