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At $z=0$ : On $D_0$ :

$$f(z) = \frac{z}{1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}\cdots\right)}$$

$$= \frac{z}{\left(\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\frac{z^8}{8!}+\cdots\right)}$$

$$= \frac{1}{z\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}\cdots\right)}$$

$$= \frac{1}{z}\,h(z)$$

where $h(z) = \frac{1}{\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\frac{z^6}{8!}\cdots\right)}$

3A,M $h(0) \overset{=2}{\neq} 0$ and $h(z)$ is analytic at $z=0$ $\therefore f$ has a simple pole at $z=0$ ✓

$\boxed{\frac{7}{8}}$

b) i) $f(z) = \left(1-\frac{1}{z^2}\right)\cosh\left(\frac{1}{z}\right)$

2A,M $= \left(1-\frac{1}{z^2}\right)\left(1+\frac{1}{2!z^2}+\frac{1}{4!z^4}+\frac{1}{6!z^6}+\frac{1}{8!z^8}\cdots\right)$

* $= 1+\frac{1}{z^2}\left(-1+\frac{1}{2!}\right)+\frac{1}{z^4}\left(\frac{-1}{2!}+\frac{1}{4!}\right)+\frac{1}{z^6}\left(\frac{-1}{4!}+\frac{1}{6!}\right)+\frac{1}{z^8}\left(\frac{-1}{6!}+\frac{1}{8!}\right)\cdots$

The general term is

$\frac{1}{z^{2n}}\left(\frac{-1}{(2n-2)!}+\frac{1}{(2n)!}\right)$ ✓, $n \geq 1$

We can also write

3A $f(z) = 1+\sum_{n=1} \frac{1}{z^{2n}}\left(\frac{-1}{(2n-2)!}+\frac{1}{(2n)!}\right)$ ✓

ii) $\frac{1}{z}$ is analytic on $\mathbb{C}-\{0\}$. $\left(1-\frac{1}{z^2}\right)$ is analytic on $\mathbb{C}-\{0\}$ and cosh is entire so $\cosh\left(\frac{1}{z}\right)$ is analytic on $\mathbb{C}-\{0\}$. $f$ is the product of two functions analytic on $\mathbb{C}-\{0\}$, so is

1A analytic on $\mathbb{C}-\{0\}$ ✓ Take $r>1$ then $\{z : |z|<r, z\neq 0\}$ is a punctured open disc containing $C$ as required ✓ eg $|z|<10$

iii) The singularity at 0 is an essential singularity ✓ This is obvious