4

$$2\pi i \left(\frac{(z-2)\cos z - \sin z}{(z-2)^2}\right)_{z=-4}$$

$$= 2\pi i \left(\frac{-6\cos 4 + \sin 4}{(-4-2)^2}\right)$$

2A $= \pi i \left(\frac{\sin 4 - 6\cos 4}{18}\right)$ ✓

$\frac{10}{13}$

b) i) Liouville's Theorem states: If f is a bounded entire function then f is constant.

Cauchy's estimate: $|f^n(\alpha)| \le \frac{K n!}{r^n}$

for f analytic on a simply connected region containing the circle $\Gamma = \{z : |z-\alpha| = r\}$ with $|f(z)| \le K$ for $z \in \Gamma$

3M Let $R = \mathbb{C}$ ✓ and let $r \to \infty$, then for $\alpha \in R$

see opp × $|f'(\alpha)| \le \frac{K(1)!}{r^1} = \frac{K}{r} \to 0$

$\therefore |f'(\alpha)| = 0$ for all $\alpha \in \mathbb{C}$ $\therefore f'(\alpha) = 0$

$\therefore f$ is constant, so Liouville's

2M theorem is proved. ✓

ii) If $|f(z)| > |z|$ then $f(z) \ne 0$ for any z and $g(z) = \frac{z}{f(z)}$ is defined. Since

$|z| < |f(z)|$, $|g(z)| < 1 \Rightarrow g$ is bounded. ✓

3M Since z, $f(z)$ are entire, $z/f(z)$ is entire. $\therefore g$ is bounded and entire so by Liouville's theorem it is constant. ✓

$\therefore g(z) = \frac{z}{f(z)} = K < 1$ $|k| < 1$ (K may not be real)

$\therefore z = K f(z) \Rightarrow \frac{z}{K} = f(z)$

$\therefore f(0) = \frac{0}{K} = 0$ ie $f(0) = 0$ ✓

3A,M Contradicting that $|f(z)| > |z|$ so there is no such f ✓

iii) $|f(z)| = |2z| = |2| \times |z| > |z|$

$\frac{12}{12}$ 1A,M for $z > 0$ $f(z) = 2z$ ✓ is an entire function satisfying $|f(z)| > |z|$ for $|z| > 0$

(72/…)