

Question 4 (Unit A4) – 25 marks

- (a) Find the derivative of each of the following functions f , and specify the domain of the derivative in each case.

(i) $f(z) = \cosh(2e^{iz})$

(ii) $f(z) = z^2 \operatorname{Log} z$

[4]

- (b) (i) Use the definition of the derivative to prove that the function

$$f(z) = (\operatorname{Re} z) + i$$

is not differentiable at $2 - 3i$.

- (ii) Use the Cauchy–Riemann equations to show that the function

$$f(x + iy) = (3x^2 - 3y^2 - y) + (6xy + x)i$$

is entire.

[9]

- (c) The function f is defined by

$$f(z) = \frac{2z + i}{z - 2i}.$$

- (i) Show that f is conformal.

- (ii) Describe the geometric effect of f on a small disc centred at $3i$.

- (iii) Show that the paths

$$\Gamma_1 : \gamma_1(t) = t^2 + i(t + 3) \quad (t \in [-2, 2])$$

$$\Gamma_2 : \gamma_2(t) = 1 + 3i + e^{it} \quad (t \in [0, 2\pi])$$

are smooth and meet at the point $3i$.

- (iv) Determine the angle from the path $f(\Gamma_1)$ to the path $f(\Gamma_2)$ at the point $f(3i)$.

[12]