Question 2 (*Unit A2*) – 25 marks

(a) Let $f(z) = \frac{1}{2}i(z-1)$.

(i) Describe the effect f has on a typical point of $\mathbb{C}$ in terms of geometric transformations.

(ii) Let Γ be the path with parametrization

$$\gamma(t) = 1 + 2e^{it} \quad \left(t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right).$$

Sketch Γ, indicating the direction of increasing t, and identifying its initial and final points in Cartesian form.

(iii) Applying your geometric description of the function f to Γ, sketch the path $f(\Gamma)$. Indicate the direction of $f(\Gamma)$ and identify its initial and final points in Cartesian form.

(iv) Write down the standard parametrization of $f(\Gamma)$. [10]

(b) The functions f, g and h are defined by

$$f(z) = \exp(iz - 3)$$
$$g(z) = \operatorname{Log}(iz - 3)$$
$$h(z) = (iz - 3)^{4i}.$$

(i) Write down the domain of each of the functions f, g and h.

(ii) Let $\alpha = -1 - 2i$. Evaluate each of the function values $f(\alpha)$, $g(\alpha)$, $h(\alpha)$. (You may give your answers in Cartesian or polar form.)

(iii) Explain why the function f does not have an inverse function.

(iv) Show that g has an inverse function and find its domain and rule. [15]

Question 3 (*Unit A3*) – 25 marks

(a) For each of the following sequences, determine whether the sequence converges, and, if it does, give the limit.

(i) $\left\{\dfrac{i^n}{\sqrt{n+2}}\right\}$

(ii) $\left\{\sin\left(\dfrac{\pi i^n}{2}\right)\right\}$ [6]

$\mathbb{C} - \{iz - 3 \le 0\}$

(b) (i) Write down the domain A of the function

$$f(z) = \frac{4+z^2}{2z+iz^2}$$

and show that $2i$ is a limit point of A.

(ii) Evaluate

$$\lim_{z \to 2i} \cosh\left(\frac{4+z^2}{2z+iz^2}\right),$$

justifying your method fully. [9]

(c) Let $B = \{z : |z| \le 2, \operatorname{Re} z > 0\}$.

(i) Sketch the sets ∂B and $\operatorname{ext} B$ on separate diagrams, and state for each set whether or not it is a region and/or compact.

(ii) Prove that the function $f(z) = \dfrac{\sin(z^2)}{1+e^{iz}}$ is bounded on the set ∂B.

(iii) Let $g(z) = ze^{-iz}$. Find points α and β in ∂B such that

$$|g(\alpha)| \le |g(z)| \le |g(\beta)|, \qquad \text{for } z \in \partial B.$$ [10]