Question 4 (*Unit A2*) – 15 marks

The functions f, g and h are defined by

$$f(z) = \exp(z^2 - 2),$$
$$g(z) = \mathrm{Log}(z^2 - 2),$$
$$h(z) = (z^2 + 2)^i.$$

(a) Write down the domain of each of the functions f, g and h. [3]

(b) Let $\alpha = 1 + i$. Evaluate each of the function values $f(\alpha)$, $g(\alpha)$, $h(\alpha)$, giving your answers in Cartesian form. [5]

(c) (i) Solve the equation

$$\mathrm{Log}(z^2 - 2) = 0.$$

(ii) Explain briefly why the function g does not have an inverse function.

(iii) Show that the restriction of the function g to the set $\{z : \mathrm{Re}\, z > 0\}$ does have an inverse function. [7]

Question 5 (*Unit A3*) – 15 marks

(a) For each of the following sequences, determine whether the sequence converges, and, if it does, find the limit.

(i) $\left\{\dfrac{1}{n^2}\left(\dfrac{1-i}{\sqrt{2}}\right)^n\right\}$

(ii) $\{\cos(\pi i^n)\}$ [6]

(b) (i) Write down the domain A of the function

$$f(z) = \frac{z^2 + 9}{z - 3i}$$

and show that $3i$ is a limit point of A.

(ii) Evaluate

$$\lim_{z \to 3i} \sqrt{\frac{z^2 + 9}{z - 3i}},$$

giving your answer in Cartesian form, and justifying your method. [9]

Question 6 (*Unit A3*) – 10 marks

Let $A = \{z : |z| \geq 2\}$.

(a) Sketch on separate diagrams the sets

∂A and $\mathrm{ext}\, A$,

and state for each of these sets whether or not it is

(i) a region,

(ii) compact. [4]

(b) Prove that the function $f(z) = \dfrac{1}{1 + e^z}$ is bounded on the set ∂A. [3]

(c) Let $g(z) = ze^z$. Find points α and β in ∂A such that

$$|g(\beta)| \leq |g(z)| \leq |g(\alpha)|, \qquad \text{for } z \in \partial A.$$ [3]