This TMA covers *Units GR5, GR6, GE5* and *GE6*.

Each question is marked out of 25. You should answer all four questions.

Question 1 (*Unit GR5*)

(a) Let G be a group of order 80.

(i) Find the possible numbers of Sylow 2-subgroups of G and of Sylow 5-subgroups of G. [5]

(ii) Using a counting argument, show that G must have *either* a unique Sylow 2-subgroup *or* a unique Sylow 5-subgroup. [5]

(iii) Deduce that G cannot be simple. [2]

(b) A group G is of order 4225.

(i) Prove that G must be the internal direct product of two of its Sylow subgroups. [5]

(ii) Show that G is necessarily Abelian. [3]

(iii) Classify all groups of order 4225. [5]

Question 2 (*Unit GR6*)

(a) Consider the group S_{11} of permutations on the set $\{1, 2, \ldots, 11\}$.

(i) Show that any element of S_{11} which is of order 11 must be an 11-cycle. [2]

(ii) Show that any element of S_{11} which is of order 5 must be *either* a 5-cycle *or* a product of *two* disjoint 5-cycles. [3]

(b) Let G be any group of order 55. Write down the possible numbers of Sylow 5-subgroups and the possible numbers of Sylow 11-subgroups of G. [3]

(c) Suppose now that G is a *non-cyclic* group of order 55.

(i) Show that, of the possibilities you found in part (b) for the numbers of Sylow 5-subgroups and Sylow 11-subgroups of G, only one can now occur; which one is it? [3]

(ii) Using the conjugacy action of G on its Sylow 5-subgroups, define a homomorphism $\phi : G \longrightarrow S_{11}$. [3]

(iii) With the help of the Orbit-stabilizer Theorem, prove that ϕ is in fact an isomorphism from G to some subgroup K of S_{11}. [5]

(iv) Let g be an element of G of order 5. Show that there is exactly one Sylow 5-subgroup H of G such that $gHg^{-1} = H$. Hence show that $\phi(g)$ is a product of two disjoint 5-cycles. [6]