This TMA covers *Units GR5, GR6, GE5* and *GE6*.

Each question is marked out of 25. You should answer all four questions.

In answering Question 1, you may *assume* that the following generalization of Theorem 4.1 of *Unit GR5* is valid:

If H_1, H_2 and H_3 are subgroups of a finite group G, then the mapping

$$\begin{aligned}\phi : H_1 \times H_2 \times H_3 &\longrightarrow G\\ (h_1, h_2, h_3) &\longmapsto h_1h_2h_3\end{aligned}$$

is an isomorphism provided that all three of the following conditions hold:

(a) $|G| = |H_1| \times |H_2| \times |H_3|$;

(b) $|H_1|$, $|H_2|$ and $|H_3|$ are coprime in pairs (that is, each pair is coprime);

(c) H_1, H_2 and H_3 are normal subgroups of G.

Question 1 (*Unit GR5*)

(a) Let G be a group of order 11011.

(i) For each prime p that divides $|G|$, find the possible numbers of Sylow p-subgroups of G; give the results of *all* the modular arithmetic calculations that need to be done in order to achieve this result. [*10*]

(ii) Show that G is the internal direct product of three of its Sylow p-subgroups. [*3*]

(iii) Explain carefully why G is Abelian. [*4*]

[In addition to Lemmas and Theorems, you may assume the results of any exercises in *Units GR1–GR5*.]

(b) Let G be a group of order 992 ($= 2^5 \times 31$). Show that G is not simple. [*8*]

Question 2 (*Unit GR6*)

(a) In this part of the question we give you five assertions, each about groups of a particular order. For each statement, say whether or not it is true. If it is true, prove it; if it is false, give a suitable counterexample (explaining why it *is* a counterexample).

You may find the following list of groups helpful as a source of counterexamples.

$$S_3, D_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_4, A_4, D_6, \mathbb{Z}_{15}, D_9.$$

(i) If a group G has order 9, then G is Abelian. [*4*]

(ii) If a group G has order p^3, where p is a prime, then G is Abelian. [*3*]

(iii) If p and q are distinct primes, then there is a non-Abelian group G of order pq. [*3*]

(iv) If G is a group of order 12, then G contains an element of order 6. [*3*]

(v) If G is a group of order 18, then G contains an element of order 3. [*4*]

(b) Let G be a *non-Abelian* group of order 689.

(i) Show that G cannot be simple. [*4*]

(ii) Show that, in the action of G by conjugation on the set of its Sylow 13-subgroups, each such subgroup is its own stabilizer. [*4*]