and H_2 is cyclic since $|H_2| = 5$, prime so $H_2 \cong \mathbb{Z}_5$ $\therefore G \cong \mathbb{Z}_3 \times \mathbb{Z}_5$ so G ✓

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is Abelian $\therefore$ G is not non Abelian $\therefore$ there is no non Abelian group of order 15. ✓ 3/3

False ✓ iv) $A_4 = \{e, (123), (132), (124), (142), (134), (143), (234), (243), (12)(34), (13)(24), (14)(23)\}$
A_4 is a group of order 12, but has elements of orders 1, 3, 2 only. ✓ 3/3

True ✓ v) If G has order 18 then by Sylows theorem G has subgroups orders 2, 3, 9. Since G has a subgroup of order 3, and 3 is prime, this subgroup is cyclic $\therefore$ it is generated by an element of order 3, which is an element of G $\therefore$ G has an element of order 3. ✓ 4/4

b) i) $|G| = 689 = 13 \cdot 53$ ✓
The divisors of $|G|$ are 1, 13, 53, 689
Consider the Sylow 13 and Sylow 53 subgroups respectively.
$p_1 = 13$ $m_1 \equiv 1 \pmod{13}$
$1 \equiv 53 \equiv 1 \pmod{13}$ and $13 \equiv 689 \equiv 0 \pmod{13}$
$\therefore m_1 = 1$ or 53 ✓

$p_2 = 53$ $m_2 \equiv 1 \pmod{53}$
$53 \equiv 689 \equiv 0 \pmod{53}$ and $13 \equiv 13 \pmod{53}$
$\therefore m_2 = 1$ ie there is a unique Sylow 53 subgroup $\therefore$ G is not simple ✓ 4/4

ii) If $m_1 = 1$, then since
1) $|G| = 13 \cdot 53 = |H_1| \cdot |H_2|$
where H_1 and H_2 are unique hence normal Sylow p-subgroups.
2) $|H_1| = 13$ and $|H_2| = 53$ are coprime
Then G is the internal direct product of its Sylow p-subgroups, and since $|H_1|, |H_2|$ are prime
$H_1 \cong \mathbb{Z}_{13}$, $H_2 \cong \mathbb{Z}_{53}$
$\therefore G \cong \mathbb{Z}_{13} \times \mathbb{Z}_{53} \Rightarrow$ G Abelian ✓
This is a contradiction $\therefore m_1 = 53$ ✓
Take a sylow 13 subgroup H_i