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1) A group G is simple if it has no non trivial proper normal subgroups.
$|G| = 992 \Rightarrow$ divisors of $|G|$ are 1, 2, 4, 8, 16, 31, 32, 62, 124, 248, 496, 992
Consider the Sylow 2-subgroups of G.
$m_1 \equiv 1 \pmod 2$, $m_1 \mid |G| \Rightarrow m_1 = 1, 31$ ✓ ✓ ½
If $m_1 = 31$ then each Sylow 2-subgroup (of order 32) has (31) non identity elements.
Consider the Sylow 31-subgroups
$m_2 \equiv 1 \pmod{31}$ $m_2 \mid |G| \Rightarrow \underline{m_2 = 1, 32}$ ✓ ½
ie $m_1 = 1$ or 31
$m_2 = 1$ or 32
Suppose $m_1 = 31$ and $m_2 = 32$
The Sylow 2 subgroups contain in total $31 \cdot 31 = 961$ non identity elements
and the Sylow 31 subgroups contain in total $32 \cdot 30 = 960$ non identity elements.
There is one identity element, but
$961 + 960 + 1 = 1922 > 992$ ∴ at least one of $m_1, m_2 = 1$. ∴ G has a proper normal subgroup ∴ G is not simple.

P.J.O

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2) a) True. To see this consider the class equation of G. Why not just use Result 5.2 of GR4?
$9 = 1 + n_2 + n_3 + \dots n_k$
Each $n_i \mid 9$ and if each $n_i = 1$ then G is Abelian. Since $n_i \mid 9$, $n_i = 1, 3$ or 9. But $n_i \neq 9$ for any i since we would have $9 = 1 + 9 = 10 \leq 1 + n_1 + \dots + n_k$ ie $9 = 10 \leq 1 + n_1 + \dots + n_k$ which is a contradiction ∴ Each $n_i = 1$ or 3.
Suppose only $n_1 = 1$. The class eqtn becomes
$9 = 1 + 3x$
$8 = 3x$
Which is impossible for x an integer.
∴ the number of ones is greater than one. Suppose the number of ones is two. The class eqtn becomes
$9 = 1 + 1 + 3x$
$7 = 3x$
Which is again impossible for x an integer ∴ the no of ones is at least