

This TMA covers Units GR3, GR4, GE3 and GE4.

Each question is marked out of 25. You should answer all four questions.

**Question 1 (Unit GR3)**

(a) Let  $A$  be the Abelian group defined by

$$A = \langle a, b, c : 8a - 3b - 6c = 0, 16a + 3b - 6c = 0, 8a - 6b + 12c = 0 \rangle.$$

(i) Write down the integer matrix corresponding to this presentation. [1]

(ii) Use row and column operations, as described in Unit GR3, to reduce your matrix to diagonal form. [9]

(iii) Hence identify  $A$  as being isomorphic to some direct product of non-trivial cyclic groups. [3]

(b) Compute the torsion coefficients and rank of each of the following Abelian groups, and state (giving a reason) whether or not they are isomorphic to each other.

$$B = \mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{175} \times \mathbb{Z},$$

$$C = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} / \langle (56, 0, 0, 0), (0, 0, 9, 0), (0, 0, 0, 25) \rangle. \quad [6]$$

(c) Let  $M = [m_{ij}]$  be a  $3 \times 3$  matrix with integer entries and let  $T$  be the matrix

$$T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let  $D$  and  $E$  be the Abelian groups whose presentations have the matrices  $M$  and  $MT$  respectively. By calculating  $MT$ , or otherwise, explain carefully why  $D$  and  $E$  are isomorphic. [6]

**Question 2 (Unit GR4)**

(a) This part concerns all Abelian groups of order 12600.

(i) Find all Abelian groups of order 12600, expressing each in canonical form. [11]

(ii) How many of the groups that you found in part (a)(i) have a cyclic subgroup of order 1400? Explain your answer carefully. [5]

(b) This part concerns the dicyclic group of order 20 given by the presentation

$$G = \langle a, b : a^5 = e, b^4 = e, ba = a^4b (= a^{-1}b) \rangle.$$

The elements of  $G$  can be written in standard form as

$$a^i b^j, \quad i = 0, 1, \dots, 4, \quad j = 0, \dots, 3.$$

The conjugacy classes of  $G$  are

$$\{e\}, \{a, a^4\}, \{a^2, a^3\}, \{b^2\}, \{ab^2, a^4b^2\}, \{a^2b^2, a^3b^2\}, \\ \{b, ab, a^2b, a^3b, a^4b\}, \{b^3, ab^3, a^2b^3, a^3b^3, a^4b^3\}.$$

(Note: you are not asked to prove this.)

(i) Write down the class equation for  $G$ . [1]

(ii) Find the centre of  $G$ . [1]

(iii) Find normal subgroups of  $G$  of orders 2 and 5. [3]

(iv) Does  $G$  possess a normal subgroup of order 10? If so, find it; if not, explain why not. [4]