

This TMA covers Units GR3, GR4, GE3 and GE4.

Each question is marked out of 25. You should answer all four questions.

Question 1 (Unit GR3)

(a) Let A be the Abelian group

$$A = \langle a, b, c : 7a - 3b - 2c = 0, 8a - 4b - 4c = 0, 2a + 2b + 8c = 0 \rangle.$$

(i) Write down the integer matrix corresponding to this presentation. [1]

(ii) Use row and column operations as described in Unit GR3, to reduce your matrix to diagonal form. [9]

(iii) Hence identify A as isomorphic to some direct product of non-trivial cyclic groups. [3]

(b) Compute the torsion coefficients and rank of each of the following Abelian groups, and state (giving a reason) whether they are isomorphic to each other:

$$B = \mathbb{Z}_{12} \times \mathbb{Z}_{22} \times \mathbb{Z}_{25} \times \mathbb{Z}_{27} \times \mathbb{Z};$$

$$C = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} / \langle (0, 44, 0, 0), (0, 0, 54, 0), (0, 0, 0, 75) \rangle. \quad [6]$$

(c) Let $\mathbf{P} = [p_{ij}]$ be a 3×3 matrix with integer entries, and let $\mathbf{Q} = [q_{ij}]$ be the transpose of \mathbf{P} ; that is to say, the rows of \mathbf{Q} are the columns of \mathbf{P} and vice versa.

Let D and E be the Abelian groups whose presentations have the matrices \mathbf{P} and \mathbf{Q} respectively. By comparing the effects of row operations on \mathbf{P} and column operations on \mathbf{Q} , and vice versa, show that D and E are isomorphic. [6]

Question 2 (Unit GR4)

(a) This part of the question concerns the Abelian group

$$A = \mathbb{Z}_{60} \times \mathbb{Z}_{72}.$$

(i) Find the primary decomposition of A . [3]

(ii) Hence write down the p -primary components of A for each prime p dividing the order of A . [3]

(iii) Find four non-isomorphic groups of order 720 (expressed as products of their primary components), each isomorphic to a subgroup of A . [8]

(iv) Why is there no subgroup of A isomorphic to \mathbb{Z}_{720} ? [2]

(b) The dicyclic group of order 12, which we shall denote here by G , has the presentation

$$G = \langle a, b : a^3 = e, b^4 = e, ba = a^2b \rangle,$$

and may be written as

$$G = \{a^m b^n : m = 0, 1, 2, n = 0, 1, 2, 3\}.$$

Its conjugacy classes are

$$\{e\}, \{b^2\}, \{a, a^2\}, \{ab^2, a^2b^2\}, \{b, ab, a^2b\}, \{b^3, ab^3, a^2b^3\}.$$

(You are not asked to prove this.)

Find a normal subgroup of G of order 3, justifying why it is a normal subgroup, and show that G has no normal subgroup of order 4. [9]