This TMA covers *Units GR1, GR2, GE1* and *GE2*.

Each question is marked out of 25. You should answer all four questions.

Question 1 (*Unit GR1*)

(a) (i) Use the Euclidean algorithm to compute hcf$\{990, 462\}$.

(ii) Use the method of back substitution to express hcf$\{990, 462\}$ as an integer combination of 990 and 462. [*12*]

(b) Let a and b be positive integers, and let p be a prime.

(i) Show that if $p \mid \text{hcf}\{a, b\}$, then p divides both a and b.

(ii) Show that if p^2 does not divide $\text{lcm}\{a, b\}$, then either p^2 does not divide a or p^2 does not divide b.

(iii) If $\text{hcf}\{a, b^2\} = p^2$, does it follow that $\text{hcf}\{a, b\} = p$? If not, give a counter-example; if so, explain why. [*13*]

Question 2 (*Unit GR2*)

(a) (i) Show that the groups

$$\mathbb{Z}_5 \times \mathbb{Z}_{50} \times \mathbb{Z}_{88}$$

and

$$\mathbb{Z}_{25} \times \mathbb{Z}_{40} \times \mathbb{Z}_{22}$$

are isomorphic.

(ii) Prove that the groups

$$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{10}$$

and

$$\mathbb{Z}_8 \times \mathbb{Z}_{15}$$

are *not* isomorphic. [*13*]

(b) Consider the matrices

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The group $G = \langle A, B \rangle$ is of order 8.

(i) Write down all 8 elements of G.

(ii) G must be isomorphic to one of the groups C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_4 and Q: find which one, and give convincing reasons for your answer. [*12*]