This TMA covers *Units GR1, GR2, GE1* and *GE2*.

Each question is marked out of 25. You should answer all four questions.

Question 1 (*Unit GR1*)

(a) (i) Use the Euclidean algorithm to calculate $\text{hcf}\{315, 1089\}$. [5]

(ii) Use the method of back substitution to express $\text{hcf}\{315, 1089\}$ as an integer combination of 315 and 1089. [5]

(b) The integers which you obtained in part (a) should be coprime. Show that this is not accidental. That is, show that if a and b are any positive integers and $\text{hcf}\{a, b\} = ac + bd$ where c and d are integers, then c and d are coprime. [6]

(c) Show that, if $\text{hcf}\{a, b\} = h$, then $\text{hcf}\{a^2, b^2\} = h^2$, by the following steps.

(i) Show that, if $\text{hcf}\{a^2, b^2\} = H$, then $H = zh^2$ for some positive integer z. [3]

(ii) Using Lemma 5.1 of *Unit GR1*, or otherwise, show that $z = 1$. [6]

Question 2 (*Unit GR2*)

(a) Show that the groups

$$\mathbb{Z}_5 \times \mathbb{Z}_{75} \times \mathbb{Z}_{63}$$

and

$$\mathbb{Z}_{45} \times \mathbb{Z}_{525}$$

are isomorphic. [5]

(b) Decide whether each of the following groups is cyclic or non-cyclic, giving reasons for your answers.

$$A = \mathbb{Z}_{63} \times \mathbb{Z}_8$$
$$B = \mathbb{Z}_{14} \times \mathbb{Z}_{36}$$
$$C = \mathbb{Z}_7 \times \mathbb{Z}_8 \times \mathbb{Z}_9$$
$$D = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_{42}.$$ [8]

(c) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in the multiplicative group of non-singular 3×3 matrices with entries in $\mathbb{R}$. Let G be the group $\langle A, B \rangle$, which you may assume is of order 8.

(i) Write down the eight elements of G, both as explicit 3×3 matrices and as products involving $\mathbf{A}$ and $\mathbf{B}$. [8]

(ii) Determine whether $\mathbf{A}$ and $\mathbf{B}$ commute. [1]

(iii) As G has order 8, it is isomorphic to one of the groups

$$C_8,\ C_2 \times C_4,\ C_2 \times C_2 \times C_2,\ D_4,\ Q.$$

Decide which one, giving reasons for your answer. [3]