This TMA covers *Units GR1, GR2, GE1* and *GE2*.

Each question is marked out of 25. You should answer all four questions.

Question 1 (*Unit GR1*)

(a) (i) Use the Euclidean Algorithm to calculate $\mathrm{hcf}\{956, 424\}$. [6]

(ii) Use the method of back substitution to express $\mathrm{hcf}\{956, 424\}$ as an integer combination of 956 and 424. [6]

(b) Let G be a group, and let h and k be elements of G with orders a and b respectively.

(i) Explain why $\langle h\rangle \cap \langle k\rangle$ is a cyclic subgroup of G. (You may find it useful to look at the *statements* of the theorems in Section 4 of *Unit GR2*; they are on page 25 of the Handbook.) [3]

Let x be a generator for $\langle h\rangle \cap \langle k\rangle$.

(ii) Show that the order of x divides both a and b. What is the maximum possible order of x? [4]

(iii) Give an example in which the order of x is the maximum possible (relative to your choice of a and b). [3]

(iv) Give an example in which the order of x is *less* than the maximum possible (relative to your choice of a and b). [3]

Question 2 (*Unit GR2*)

(a) (i) Show that the groups

$$\mathbb{Z}_9 \times \mathbb{Z}_{16} \quad \text{and} \quad \mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_8$$

are *not* isomorphic. [6]

(ii) Show that the groups

$$\mathbb{Z}_8 \times \mathbb{Z}_{15} \times \mathbb{Z}_{63} \quad \text{and} \quad \mathbb{Z}_{45} \times \mathbb{Z}_{168}$$

are isomorphic. [7]

(b) Let G be the group of matrices generated by

$$a = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Given that G is a group of order 8, to which of the groups

$$C_8,\ C_2 \times C_4,\ C_2 \times C_2 \times C_2,\ D_4,\ Q$$

is G isomorphic? Explain the reasons for your answer carefully. [12]