Question 3 — 10 marks

(a) Determine whether each of the following sequences $\{a_n\}$ is convergent, stating the limit of the sequence (if it exists).

(i) $$a_n = \frac{2n^4 + 3(n!) + 5^n}{4n^4 - 2(n!) + 3^n}$$ [2]

(ii) $$a_n = \frac{4n^3 + 2^n - 3}{2n^3 + 8n^2 + n - 1}$$ [3]

(b) Determine whether each of the following series is convergent.

(i) $$\sum_{n=1}^{\infty} \frac{2n}{n^3 - 4n + 5}$$ [2]

(ii) $$\sum_{n=1}^{\infty} \frac{n + \sin n}{4^n}$$ [3]

You should name any results or tests that you use.

Question 4 — 10 marks

Let f be the function defined by

$$f(x) = \begin{cases} x^3, & x < 0; \\ \sin 2x, & 0 \le x \le \pi; \\ e^{-x}, & x > \pi. \end{cases}$$

(a) Sketch the graph of f. [2]

(b) Prove that f is discontinuous at π. [2]

(c) Prove that f is continuous on $\mathsf{R} - \{\pi\}$. [6]

Question 5 — 10 marks

The set $G = \{1, 5, 7, 11, 13, 17, 19, 23\}$ forms a group under multiplication modulo 24. (You are NOT asked to prove this statement.)

(a) What are the possible orders of subgroups of G? For each possible order either find a corresponding subgroup or justify that none exists. [5]

Denote by H the subgroup of order 2 that you found in part (a).

(b) Show that the subgroup H is normal in G. [1]

(c) List the elements of the quotient group G/H and identify this quotient group up to isomorphism. (That is, identify a modular arithmetic group or a group of symmetries of some figure to which the quotient group is isomorphic.) [4]

Question 6 — 10 marks

The group $S(\triangle) = \{e,\ a,\ b,\ r,\ s,\ t\}$ acts on the set

$$X = \{S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_6,\ M_1,\ M_2,\ M_3,\ M_4,\ M_5,\ M_6\}$$

of locations of line segments in the figure below in the natural way. That is, for $g \in S(\triangle)$ and $x \in X$

$$g \wedge x = y,$$

where y is the location of the line segment, initially at x, after the symmetry transformation g. (You are NOT asked to prove that this is a group action.)

$2 = A(n+2) + B(n-2)$

$A = 1/2$

$2 = \frac{5n}{2} + B$

$B = -1/2$

$\frac{1}{2(n-2)} - \frac{1}{2(n+2)}$