$\lim_{n\to\infty} L(f, P_n) = \frac{1}{n^2}(2n^2 - 3n + 2) = 2 - \frac{3}{n} + \frac{2}{n^2} \to 2$ as $n \to \infty$

$\lim_{n\to\infty} U(f, P_n) = \frac{2n+1}{n} = 2 + \frac{1}{n} \to 2$ as $n \to \infty$

Hence f is integrable on $[0,1]$ and it follows that $\int_0^1 f(x) = 2$.

2/2

b) $f(x) = \frac{x^2}{(1+x)^{1/2}}$

On $[0,1]$, $(1+x)^{1/2} \geq 1$

$\therefore f(x) \leq x^2$

Also max value of $(1+x)^{1/2}$ on $[0,1]$ is $\sqrt{2}$

$$\therefore \frac{x^2}{\sqrt{2}} \leq \frac{x^2}{(1+x)^{1/2}} \leq x^2$$

$$\int_0^1 \frac{x^2}{\sqrt{2}}\,dx \leq \int_0^1 \frac{x^2}{(1+x)^{1/2}}\,dx \leq \int_0^1 x^2\,dx$$

$$\left[\frac{x^3}{3\sqrt{2}}\right]_0^1 \leq \int_0^1 \frac{x^2}{(1+x)^{1/2}}\,dx \leq \left[\frac{x^3}{3}\right]_0^1$$

$$\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{(1+x)^{1/2}}\,dx \leq \frac{1}{3}$$

4/4

c) $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$

$$I_{n+1} - I_n = \int_0^1 \frac{dx}{(1+x^2)^{n+1}} - \int_0^1 \frac{dx}{(1+x^2)^n}$$

$$= \int_0^1 \left(\frac{1}{(1+x^2)^{n+1}} - \frac{1}{(1+x^2)^n}\right)dx$$

$$= \int_0^1 \left(\frac{1}{(1+x^2)^{n+1}} - \frac{(1+x^2)}{(1+x^2)^{n+1}}\right)dx$$

$$= \int_0^1 \frac{-x^2}{(1+x^2)^{n+1}}\,dx = -\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$$

Integrate by parts

Put $u = x^2 \Rightarrow du = 2x$

$dv = (1+x^2)^{-n-1}$ $\quad v = \frac{-1}{2nx}(1+x^2)^{-n}$