All the terms in brackets are greater than or equal to $1/2$, as are 1 and $1/2$, the first two terms.

$S_{2k} > 1 + \frac{1}{2} + \frac{k}{2} + \dots\dots + \frac{1}{2} = 1 + \frac{k}{2}$

as $k \to \infty$, $S_{2k} \to \infty$, so the series is divergent.

At $x = 5/2$

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} \frac{(x-1)^n}{n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{3}\right)^n \frac{(5/2-1)^n}{(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots\dots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} \neq \sum_{n=1}^{\infty} \frac{1}{(2n-1)2n} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

But $\sum_{n=1}^{\infty} 1/n^2$ is convergent. Hence by the comparison test, the series is convergent at $x = 5/2$

The interval of convergence is therefore $[-1/2, 5/2]$

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$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} \frac{(x-1)^{n+1}}{(n+1)^2}$ has radius of convergence $3/2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1} \frac{2^{n+1}}{3^{n+1}} \frac{1}{(n+2)^2} \Big/ (-1)^n \frac{2^n}{3^n} \frac{1}{(n+1)^2}\right|$$

$$= \frac{2(n+1)^2}{3(n+2)^2} \to \frac{2}{3} \text{ as } n \to \infty$$

Careful $\left\{ a_n = \frac{(-1)^{n-1} 2^{n-1}}{3^{n-1} \cdot n^2} \right.$

The radius of convergence is given by the reciprocal of this limit ie $1/(2/3) = 3/2$

Simpler to use Integration Rule

c) $f(x) = (1 + 3x^5)^{7/11}$

$$(1+t)^n = 1 + nt + \frac{n(n-1)t^2}{2!} + \frac{n(n-1)(n-2)t^3}{3!} \dots\dots$$

Put $t = 3x^5$, $n = 7/11$