(iii) Find the equations of the lines representing the tangents you found in parts (b)(i) and (ii) in the standard embedding plane $z = 1$. Give a geometrical interpretation of these lines in relation to the conic E in part (a). [4]

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(Analysis Block B)

Question 1 (Unit 1) – 25 marks

(a) (i) Sketch the graph of the function f, where

$$f(x) = \begin{cases} 3x - 2, & x < 1; \\ x^3, & x \geq 1. \end{cases} \qquad [2]$$

(ii) Use the Strategy in Frame 4 of Section 1 to decide whether f is differentiable at the point 1. If it is, calculate $f'(1)$. [5]

(b) (i) Prove that the function $g(x) = x^7 + x^3 + x + 1$ has an inverse function g^{-1} that is differentiable on $\mathbb{R}$. (You may assume that the image $g(\mathbb{R})$ is $\mathbb{R}$.) [6]

(ii) Show that $g^{-1}(4) = 1$, and determine $(g^{-1})'(4)$. [2]

(c) Determine whether the following limit exists, and if it does exist, find its value:

$$\lim_{x \to 1} \frac{x - 4x^4 + 3x^5}{(x-1)^2}. \qquad [10]$$

Question 2 (Unit 2) – 25 marks

(a) Let f be the function

$$f(x) = \begin{cases} 1 + 2x, & 0 \leq x < 1; \\ 1, & x = 1. \end{cases}$$

(i) Sketch the graph of f. [1]

(ii) Using the standard partition P_n of $[0, 1]$, where $n > 2$, show that

$$L(f, P_n) = \frac{2n^2 - 3n + 2}{n^2} \quad \text{and} \quad U(f, P_n) = \frac{2n+1}{n}. \qquad [9]$$

(iii) Deduce that f is integrable on $[0, 1]$ and evaluate

$$\int_0^1 f. \qquad [2]$$

(b) Prove that

$$\frac{1}{3\sqrt{2}} \leq \int_0^1 \frac{x^2}{(1+x)^{\frac{1}{2}}}\, dx \leq \frac{1}{3}. \qquad [4]$$

(c) Let

$$I_n = \int_0^1 \frac{dx}{(1+x^2)^n}, \quad n = 1, 2, \ldots.$$

(i) By considering $I_{n+1} - I_n$ and using Integration by Parts, show that

$$2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n. \qquad [6]$$

$2n(I_{n+1} - I_n) = \frac{1}{2^n} - I_n$

(ii) Determine the value of I_3. [3]

$I_3 = \int_0^1 \frac{1}{(1+x^2)^3}\, dx$

$I_{n+1} - I_n = \frac{1}{2n2^n} - \frac{I_n}{2n}$

$= \frac{1}{n2^{n+1}} - \frac{I_n}{2n}$