MST209

Assignment Booklet I

Contents **Cut-off date**

2	TMA MST209 01 Part 1 (covering *Unit 1*)	7 November 2012
4	TMA MST209 01 Part 2 (covering *Units 2*, *3* and *4*)	28 November 2012
8	TMA MST209 02 (covering *Units 5*, *6* and *7*)	19 December 2012
13	TMA MST209 03 (covering *Units 8*, *9*, *10* and *11*)	30 January 2013
18	CMA MST209 41 (covering *Units 1–11*)	6 February 2013

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

You will find instructions on how to fill in the PT3 form in the current *Assessment Handbook*. Remember to fill in the correct assignment number as listed above. Remember also to allow sufficient time in the post for the TMA to reach your tutor on or before the cut-off date. Marks allocated to each part of each TMA question are indicated in brackets in the margin.

You must submit answers to the computer-marked assignment (CMA) electronically, from your StudentHome web page, by the cut-off date. The *Assessment Handbook for Undergraduate Modules* (accessible via StudentHome) states that you should make sure that you have submitted your CMA online by midday (UK local time) on the cut-off date. However, there is a 12-hour grace period, so any CMAs received before midnight will still be accepted; but we strongly recommend that you do not leave submission of your CMA to the last minute. We also recommend that you keep all submission receipts.

For some assignment questions you will be directed to submit Mathcad printout as part of your solution. Please submit only the pages that you are directed to submit, and please annotate and/or highlight the significant parts (i.e. your input and the results that you are using). See the *MST209 Guide* for further details.

WEB 03090 0
9.1

Question 1 below, on *Unit 1*, forms the first part of TMA MST209 01. The remainder of the TMA (Part 2, on *Units 2*, *3* and *4*) can be found immediately following Question 1 in this booklet. Question 1 is marked out of 25. (The whole TMA is marked out of 100.)

In order to encourage you to present your solutions to the TMA questions in a good mathematical style, your tutor will comment on how you:

- use correct mathematical notation;
- define any symbols that you introduce in formulating and solving a problem;
- give references for standard formulae and derivations;
- include comments and explanations within your mathematics;
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question;
- draw diagrams and graphs;
- annotate your Mathcad worksheets.

These features are seen as being essential to complementing your mathematical skills. Your tutor will make comments on how well you achieve these objectives and give you guidance on how to satisfy the threshold requirement.

Five of the marks for TMA 01 Part 2, and a similar amount in later TMAs, will be allocated to the way you write your solutions. It is expected that most students will receive the majority of the presentation marks; such marks are included in TMAs to encourage, and emphasize the need for, thinking about how you present your mathematics.

Please send your answers to Question 1 to your tutor, along with a TMA form (PT3). Be sure to fill in the assignment number on this form as

$\boxed{\text{MST209 01}}$.

Your tutor will mark and comment on your answer to Question 1, and will send it back to you directly to give you some early feedback on your work, and on your mathematical style. He or she will retain your PT3 to enter on it your marks for the rest of this assignment. Your copy of the form will be returned to you, via Walton Hall, with your answers to Part 2 of this assignment. Do *not* send another PT3 form with Part 2 of this TMA.

Question 1 (*Unit 1*) – 25 marks

In parts (a), (b)(i), (c) and (d)(i) you must not use your computer to solve the problems, though you may of course use it to check your answers. The solutions that you submit to your tutor must show your working, and contain explanations of how you obtained your answers. You may, however, quote any formulae from the MST209 Handbook that you need, provided that you reference it clearly.

In parts (b)(ii), (b)(iii), (d)(ii) and (d)(iii) you are expected to use your computer. We have suggested that you use certain Mathcad worksheets associated with *Unit 1*; you are not obliged to do so, but you will find that these worksheets are convenient because, when using them, you can consult the 'Mathcad tips' pop-ups if you can't remember what to do.

When you use Mathcad, it is important that you make your tutor aware of the results that you wish to be considered: you should not leave your tutor to interpret Mathcad printouts.

(a) Real variables x and y are related by the equation

$$\ln(3y+2) = 5\ln(x-1) - \ln(2-x) - x.$$

(i) Determine the range of values of x and y for which the expressions on each side of this equation are defined. [2]

(ii) Find y explicitly as a function of x, that is, express the equation in the form $y = f(x)$, simplifying your answer as far as possible. [3]

(b) (i) Show that the expressions $2\sqrt{3}\sin(3t + \pi/3)$ and $\sqrt{3}\sin(3t) + 3\cos(3t)$ are equivalent. [2]

(ii) Use Mathcad to graph the function $f(t) = \sqrt{3}\sin(3t) + 3\cos(3t)$ on the interval $-\pi \leq t \leq \pi$. On the same graph, plot the function $g(t) = 0.8t^2$. How many solutions does the equation $f(t) = g(t)$ have on the interval $-\pi \leq t \leq \pi$? Briefly justify your answer. [3] PC

(iii) Use Mathcad to find, correct to four decimal places, all of the solutions of the equation $f(t) = g(t)$ on the interval $-3 \leq t \leq -1$. (You can use the Mathcad worksheet `20901-02 Solving numerically.xmcd`, where you will need to follow the instructions concerning solve blocks in the Mathcad tips pop-up.) Submit to your tutor just one page of Mathcad printout, on which you should clearly identify the input and your solutions. [3] PC

(c) If $xy^2 + 3x^2y^2 - x^3y = 5$, use implicit differentiation to determine dy/dx, expressing your answer in the form

$$\frac{dy}{dx} = f(x,y),$$

that is, an expression that involves terms in both x and y. [4]

(d) (i) Without using Mathcad, determine the indefinite integral of the function

$$f(x) = \frac{1}{(x+3)(x-2)} \qquad (-3 < x < 2).$$ [3]

(ii) Use Mathcad to obtain the indefinite integral in part (d)(i), and comment on the result that you obtain. (If you wish, you can use the Mathcad worksheet `20901-05 Integration.xmcd`, which has a Mathcad tips pop-up.) [2] PC

(iii) Evaluate the integral

$$\int_{\pi/2}^{2\pi/3} \sec(2x)\,dx.$$

(There is no need to evaluate your answer numerically.)

Compare your answer with that obtained using Mathcad. [3] PC

Questions 2 to 6 below, on *Units 2*, *3* and *4*, form the second part of TMA 01. Your overall grade on TMA 01 will be based on the sum of your marks on these questions and on the question in Part 1.

In order to encourage you to present your solutions to the TMA questions in a good mathematical style, there are 5 presentation marks on this TMA given for how you:

- use correct mathematical notation;
- define any symbols that you introduce in formulating and solving a problem;
- give references for standard formulae and derivations;
- include comments and explanations within your mathematics;
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question;
- draw diagrams and graphs;
- annotate your Mathcad worksheets.

These features are seen as being essential to complementing your mathematical skills. Your tutor will have made comments on how to achieve the threshold requirement for these objectives in the first part of this TMA. The presentation marks will be put in the box for Question 7 on the TMA form (PT3).

Please send your answers to Questions 2 to 6 to your tutor. Your tutor should have kept the PT3 for this assignment, so there is no need to send another. (If your tutor has returned your original PT3 by mistake with your answer to Question 1, send it back with your answers to Questions 2 to 6.) Your copy of the form will be returned to you with your answers to these questions.

Question 2 (*Unit 2*) – 14 marks

In each of parts (a) and (b) you must solve the problem by hand, and the solution that you submit to your tutor should contain all your working.

(a) Consider the differential equation

$$2y + \frac{dy}{dx}\tan(2x) = \cos(2x) \quad (\pi/4 < x < \pi/2).$$

Which of the methods of finding analytic solutions of differential equations described in *Unit 2* could you use to solve this equation? Give reasons for your answer.

Find the general solution of the differential equation, expressing y explicitly as a function of x. Hence find the particular solution of the differential equation that satisfies the initial condition $y(\pi/4) = 1$. *[8]*

(b) Consider the differential equation

$$e^t \frac{dy}{dt} = y^3 \quad (0 < y).$$

Which of the methods of finding analytic solutions of differential equations described in *Unit 2* could you use to solve this equation? Give reasons for your answer.

Find the general solution of the differential equation, expressing y explicitly as a function of t. Hence find the particular solution of the differential equation that satisfies the initial condition $y(0) = 1$. [6]

Question 3 (*Unit 2*) – 9 marks

This question is concerned with the use of Euler's method to find a numerical solution to the initial-value problem

$$\frac{dy}{dx} = 2x^2 - 3y^2, \quad y(0) = 0.$$

In part (a) you may use a computer or calculator only to perform numerical calculations. In part (b), on the other hand, you are expected to use one of the MST209 Mathcad worksheets. You may find it helpful to use the same worksheet in part (c).

(a) Use Euler's method with a step size of 0.1 to find an approximation to the value of $y(0.3)$, where $y(x)$ is the solution to the given initial-value problem. Carry out your calculations using at least five decimal places. Show all your working, and quote your final answer to four decimal places. [3]

(b) Use the Mathcad worksheet `20902-02 Euler's method.xmcd` associated with *Unit 2*, Activity 2.3, to calculate approximations to six-decimal-place accuracy to the value of $y(1)$, where $y(x)$ is the solution to the given initial-value problem, with step sizes $h = 0.01$, 0.001 and 0.0001. (You may have to edit the worksheet, by entering the appropriate right-hand side for the differential equation, the appropriate initial values and the number (three) of step sizes.) Submit to your tutor the Mathcad printouts that show your edited inputs and the output. [3] PC

(c) The value 0.526 709 of the solution $y(1)$, which is correct to six decimal places, has been obtained using a different numerical method. Using the three approximate values for $y(1)$ that you have obtained using Euler's method in part (b), confirm that 'absolute error is approximately proportional to step size' (page 80 of *Unit 2*) when Euler's method is used for this initial-value problem with step sizes $h = 0.01$, 0.001 and 0.0001. Find the constant of proportionality correct to one decimal place. PC

(*Hint*: You may find it helpful to construct a table of the following form.

Step size	Approximation	Correct value	Absolute error	$\dfrac{\text{Absolute error}}{\text{Step size}}$
0.01		0.526 709		
0.001		0.526 709		
0.0001		0.526 709		

The approximate values and absolute errors may be obtained from the Mathcad worksheet.) [2]

(d) Use your answer to part (c) to predict the size of the absolute error in calculating an approximation to $y(1)$ using a step size of 0.000 001. [1]

Question 4 (*Unit 3*) – 23 marks

In parts (a)–(c) you must solve the problem by hand, and you must show your working in your solution. In part (d) you are expected to use a computer.

(a) Determine the general solutions of the following linear second-order homogeneous differential equations.

(i) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 7y = 0$

(ii) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 16y = 0$

(iii) $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0$ [*6*]

(b) Find a particular integral of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 9e^{-4x} - 125x.$$

Hence write down the general solution of this equation. [*6*]

(c) Find the particular solution to the initial-value problem

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 9e^{-4x} - 125x, \quad y(0) = \tfrac{8}{5}, \; y'(0) = 1.$$ [*5*]

(d) Use Mathcad to plot the particular solution to the initial-value problem in part (c) for x between 0 and 2. The particular solution to the initial-value problem in part (c) can be split into terms arising from the complementary function and those from the particular integral. Use Mathcad to plot both of these functions. [*4*] PC

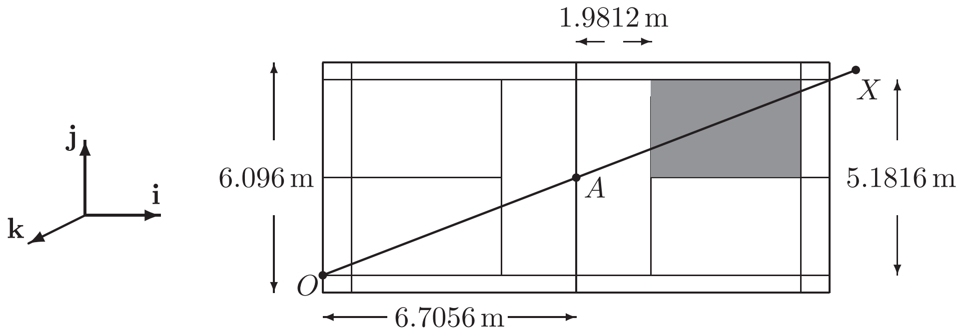
(e) Using part (d), or otherwise, identify the approximate solution to the initial-value problem in part (c) for large values of x. Give a very brief justification. [*2*]

Question 5 (*Unit 4*) – 21 marks

Note that throughout this question, vectors are shown in **bold** type (e.g. **v**) or with an over-arrow (e.g. $\overrightarrow{OA}$). When writing your solutions, if use of an over-arrow is inappropriate, then you should use underlining to show a vector quantity (see Subsection 1.2 on page 155 of *Unit 4*). If you type your assignments, then vectors must be in **bold** type. If you fail to distinguish vectors in this way, you will certainly lose some of the presentation marks available for this assignment.

In this question you should quote all numerical answers *correct to two decimal places.*

The dimensions of a badminton court are as shown below in plan view.



A player at the point O smashes the shuttlecock from a point Y at a height of 3.00 m vertically above O. Assume that the shuttlecock then travels in a straight line directly over the net at the midpoint A, which is at a height of 1.55 m, before bouncing at the point X. The unit vector $\mathbf{k}$ is directed vertically upwards.

(a) Taking the origin at O and axes as shown in the figure, write down the position vectors of the points Y and A. [2]

(b) Determine the position vector of any point on the line YA, and hence find the position vector of the point X. Deduce that the shuttlecock cannot land in the shaded area of the court. [7]

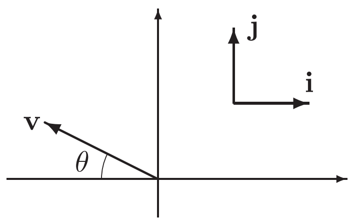
(c) Find the distance travelled by the shuttlecock between the point of the smash and hitting the floor. [2]

(d) Find the dot product of the vectors $\overrightarrow{XO}$ and $\overrightarrow{XY}$, and hence determine the angle below the horizontal at which the shuttlecock travels before landing. Give your answer in degrees correct to two decimal places. [4]

(e) Find the cross product of the vectors $\overrightarrow{YX}$ and $\overrightarrow{YO}$, and use this result to determine the area of the triangle OYX and a unit vector perpendicular to $\overrightarrow{YX}$ and $\overrightarrow{YO}$. [6]

Question 6 (*Unit 4*) – 3 marks

Consider the vector $\mathbf{v}$ shown in the following diagram.



Find the $\mathbf{i}$- and $\mathbf{j}$-components of $\mathbf{v}$ in terms of the magnitude of $\mathbf{v}$ and θ, simplifying your answers as far as possible. [3]

This assignment covers *Units 5*, *6* and *7*.

Please remember that if you write your answers, you should underline all vectors, to make it clear that they are vectors — see Subsection 1.2 on page 155 of *Unit 4*. (If you type your answers, then you should use **bold** for each vector.) Persistent failure to identify vectors in this way will lose marks.

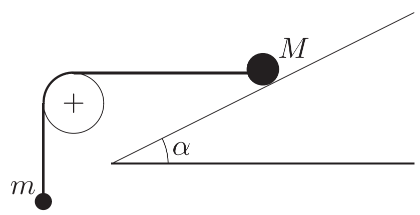
As in TMA 01, there are 5 marks awarded on this TMA for how you:

- use correct mathematical notation;
- define any symbols that you introduce in formulating and solving a problem;
- give references for standard formulae and derivations;
- include comments and explanations within your mathematics;
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question;
- draw diagrams and graphs;
- annotate your Mathcad worksheets.

The presentation marks will be put in the box for Question 5.

Question 1 (*Unit 5*) – 23 marks

A block M lies in equilibrium on a rough plane inclined at an angle α to the horizontal, and the coefficient of static friction between the block and the plane is μ. The direction of the string attached to block M is horizontal; the string passes over a model pulley and is attached to a dangling block m as shown in the diagram below.



(a) Model the blocks as particles. Draw two force diagrams showing all the forces acting on the two particles, and briefly describe the nature of each force. [4]

(b) On each force diagram, draw your choice(s) for the coordinate axes, and briefly explain why you have chosen them to be oriented in this way. [2]

(c) Express all the forces in terms of your unit vectors chosen in part (b). [5]

(d) Hence derive, with justification, scalar equations representing the fact that the two particles are in equilibrium. [3]

(e) Write down further equations representing the facts that the pulley is a model pulley and that the system is in equilibrium. [2]

(f) Show that for the system to remain in equilibrium,

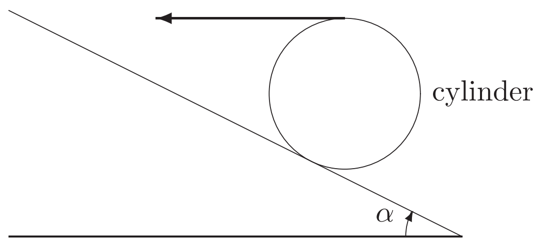
(i) $M \geq m \tan\alpha$,

(ii) $\mu \geq \dfrac{M\sin(\alpha) + m\cos(\alpha)}{M\cos(\alpha) - m\sin(\alpha)}$. [5]

(g) Give a range of values of α for which this model is valid, with justification. (Note that the validity of the model is independent of the value of the coefficient of friction.) [2]

Question 2 (*Unit 5*) – 19 marks

A uniform cylinder of mass m and radius R rests in equilibrium against a rough plane that is inclined at an angle α to the horizontal. The cylinder is supported by a cord under a constant tension, wrapped round it, so that the cord leaves the surface of the cylinder tangentially and is horizontal; the plane of the cord is perpendicular to the axis of the cylinder. The axis of the cylinder is horizontal, and all the forces act in the same vertical plane.



Model the cord as a model string, and take the coefficient of static friction between the cylinder and the plane as μ. The object of this question is to find the minimum value of the coefficient of friction for the cylinder to be in equilibrium.

(a) Draw a force diagram showing all the forces acting on the cylinder. Clearly define each of your forces. [3]

(b) Choose an appropriate coordinate system, and express each of the forces in terms of your chosen unit vectors. [4]

(c) Choose a point about which to take torques, briefly justifying your choice. [1]

(d) Write down a position vector of a point on the line of action of each force relative to the point chosen in part (c). [2]

(e) Find the torque of each force about the point chosen in part (c). By applying the equilibrium condition for torque, the equilibrium condition for forces and the friction law, show that the magnitude of the tension in the cord is

$$\frac{\sin\alpha}{1+\cos\alpha}\,mg.$$ [4]

(f) Hence find the minimum value of the coefficient of friction necessary to maintain equilibrium. [3]

(g) By considering $\alpha \to 0$, check that this condition is realistic.

Explain whether the model is valid for $\alpha > \pi/2$. [2]

Question 3 (*Unit 6*) – 30 marks

The solutions that you submit to your tutor should show all your working.

In part (a) you may derive the equation of motion or you may quote appropriate formulae. If you do quote a formula, then please give a clear reference to the MST209 Handbook, and justification for why the formula is appropriate.

A ball is projected vertically upwards with an initial speed of $10\,\mathrm{m\,s^{-1}}$ at a height of 2.5 m above the ground. In part (a) ignore all frictional forces.

(a) (i) Draw a force diagram for the ball while it is in motion. [*1*]

(ii) Define appropriate coordinate axes and an origin, and state the initial velocity and initial displacement in terms of the unit vectors and origin that you have chosen. [*2*]

(iii) Determine, in terms of the magnitude of the acceleration due to gravity, g, the maximum height that the ball reaches above the point of projection, and the time taken to reach this position. [*3*]

(iv) Determine the speed at which the ball hits the ground, correct to two decimal places, taking the value of g to be $9.81\,\mathrm{m\,s^{-2}}$. [*2*]

In the remainder of the question revise this model by taking air resistance into account. Model the ball as a sphere of diameter D and mass m, and assume that the quadratic model of air resistance applies.

(b) In this part of the question the upward motion of the ball is investigated. The ball is projected at time $t = 0$.

(i) Draw a force diagram showing all the forces acting on the ball, and express each force in terms of the unit vectors using the same axes as in part (a), justifying your derivation. [*2*]

(ii) Show that the component of acceleration at time t in the upward direction is given by

$$a = -\frac{g}{b^2}(v^2 + b^2),$$

where v is the speed of the ball at time t, and $b^2 = mg/0.2D^2$. [*2*]

(iii) By writing $a = dv/dt$, solve the resultant differential equation and determine the time t in terms of v, b, g and v_0, where v_0 is the initial speed, upwards, of the ball. [*4*]

(iv) By writing $a = v\,dv/dx$, solve the resulting differential equation and determine the height above the point of projection at time t in terms of v, b, g and v_0. [*4*]

(c) For this problem, the data values are as follows.

m	D	v_0
0.01 kg	0.02 m	$10\,\mathrm{m\,s^{-1}}$

Using your solutions from part (b), calculate the time taken for the ball to reach its maximum height, and find the maximum height of the ball above the point of projection, both correct to two decimal places.

Are these values reasonable when compared to the values found in part (a)(iii) for a model that neglected air resistance? [*4*]

(d) In this part of the question the downward motion of the ball is investigated. The origin or coordinate system may be changed, in which case they must be clearly defined.

(i) Draw a force diagram showing all the forces acting on the ball, and express each force in terms of the unit vectors, justifying your derivation. [*2*]

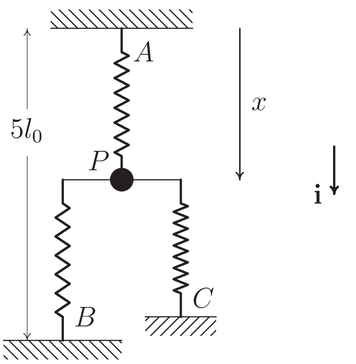
(ii) Derive the equation of motion (i.e. a differential equation) of the ball, as it moves downwards, in terms of v, b and g. Hence calculate the terminal speed of the ball if it could continue to fall indefinitely beyond the original point of projection. [*3*]

(iii) For the coordinate system that you have chosen, what is the condition to determine when the ball hits the ground? [*1*]

Question 4 (*Unit 7*) – 23 marks

You may ignore air resistance and other frictional forces in this question.

A block P of mass m is attached to three springs whose other ends are attached to fixed points A, B and C. The stiffnesses of the three springs and their natural lengths are given in the table below (see part (b)). The point B is a distance $5l_0$ below A, and the point C is a distance $\frac{1}{2}l_0$ above B. The diagram illustrates the arrangement of the springs and the block.



Model the block as a particle and the springs as model springs. Take the origin at A, with the displacement of P from A being x, so that the x-axis is as shown, and $\mathbf{i}$ is pointing downwards.

(a) Draw a force diagram indicating all the forces acting on the particle. [*2*]

(b) Copy and complete the table below to give the spring force for each spring. The notation is the same as that used in the unit.

Spring	Spring length	Natural length	Extension	Stiffness	$\widehat{\mathbf{s}}_i$	$\mathbf{H}_i$
AP		l_0		$2k$		
BP		$\frac{1}{2}l_0$		$8k$		
CP		$\frac{3}{2}l_0$		$6k$		

Express all the other forces in component form. [*5*]

(c) Derive a differential equation of motion of the particle. [*2*]

(d) Find the position of equilibrium for the particle. [*2*]

(e) Find the general solution of the differential equation found in part (c). [*4*]

(f) The particle is initially released from rest at a distance $\frac{7}{2}l_0$ below A. Determine the solution of the differential equation that satisfies these initial conditions. [*4*]

(g) Write down the period and the amplitude of the oscillations of the particle during its subsequent motion. [*2*]

(h) Draw a sketch of the graph of x against t for $t \geq 0$, clearly indicating the amplitude, period, starting position and average position of the particle. [*2*]

This assignment covers *Units 8*, *9*, *10* and *11*.

As previously, there are 5 marks awarded on this TMA for how you:

- use correct mathematical notation;
- define any symbols that you introduce in formulating and solving a problem;
- give references for standard formulae and derivations;
- include comments and explanations within your mathematics;
- explicitly state results and conclusions, giving answers to an appropriate degree of accuracy and interpreting answers in the context of the question;
- draw diagrams and graphs;
- annotate your Mathcad worksheets.

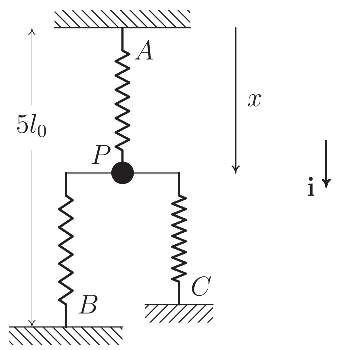
Presentation marks will be put in the box for Question 9.

Question 1 (*Unit 8*) – 14 marks

This question analyses the same system of springs and a mass that was used in TMA 02 Question 4, but now using the energy methods of *Unit 8*.

A block P of mass m is attached to three springs whose other ends are attached to fixed points A, B and C. The stiffnesses of the three springs and their natural lengths are given in the table below. The point B is a distance $5l_0$ below A, and the point C is a distance $\frac{1}{2}l_0$ above B.

If you introduce any variables not defined in the question, then you must define them.



Spring	Stiffness	Natural length
AP	$2k$	l_0
BP	$8k$	$\frac{1}{2}l_0$
CP	$6k$	$\frac{3}{2}l_0$

(a) State your choice for the datum for gravitational potential energy of particle P. [*1*]

(b) Write down the gravitational potential energy of particle P at a general point of its motion. [*1*]

(c) Write down the kinetic energy of particle P at a general point of its motion. [*1*]

(d) Determine the potential energy stored in each spring at a general point of its motion. [*4*]

(e) Write down an equation representing the total mechanical energy for the system at a general point of its motion. By differentiating this equation with respect to time, verify that your answer is equivalent to the equation of motion derived in Question 4 of TMA 02. Briefly justify your answer. [*7*]

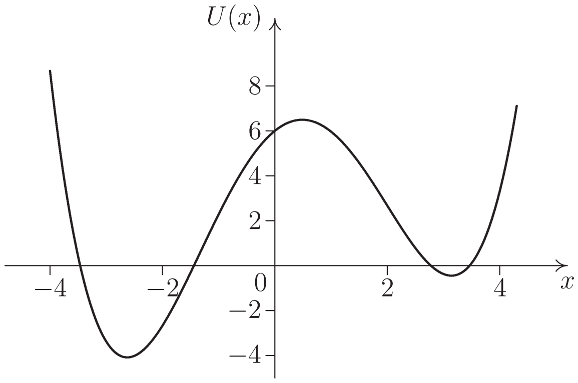
Question 2 (*Unit 8*) – 7 marks

It is recommended that you use Mathcad to answer part (b) of this question.

This question concerns a system with potential energy function given by

$$U(x) = \tfrac{1}{8}x^4 - \tfrac{1}{6}x^3 - 2x^2 + 2x + 6,$$

part of whose graph is sketched below.



(a) Write down, in the form $F(x)\,\mathbf{i}$, the force that gives rise to this potential energy function. [*1*]

(b) The total energy of the system is a constant E. For each of the following values of E state, with a reason, a range or ranges (if any), accurate to one decimal place, of x-values that could represent a motion of the system. You must explain how you derive the x-values. (If you use the same method to derive each value, then only one explanation is required.) PC

(i) $E = -5$ [*1*]

(ii) $E = -3$ [*2*]

(iii) $E = 3$ [*2*]

(iv) $E = 7$ [*1*]

Question 3 (*Unit 9*) – 8 marks

Do not use your computer to answer this question, except possibly to check your answer. Your solution must show your working.

Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 2, \\ 2x_1 + 2x_2 + x_3 &= 1, \\ x_1 - 4x_2 + 3x_3 &= 1. \end{aligned}$$

Express the system of equations in augmented form.

Use Gaussian elimination to reduce the system to upper triangular form. Carry out your calculations by hand with exact arithmetic using the methods of Subsections 1.2 and 1.3 of *Unit 9*. Clearly label the operations that you use. If the set of equations has no solution, then clearly state this; if it has a unique solution, find it; and if it has an infinite number of solutions, find the general solution. [*8*]

Question 4 (*Unit 9*) – 16 marks

Do not use your computer to answer this question, except possibly to check your answer. Your solution must show your working.

Consider the following table of data values.

i	1	2	3	4
x_i	2	4	5	6
y_i	1	3	10	33

(a) Explain which three points should be used in order to find a quadratic polynomial approximation to the value of $y(3)$. [2]

(b) Using these three data points, construct the system of equations $\mathbf{Xa} = \mathbf{y}$, which it is necessary to solve to find the quadratic approximation to $y(3)$. [4]

(c) Use Gaussian elimination to solve these equations for the coefficient vector $\mathbf{a}$, and find the quadratic polynomial approximation. Hence find the quadratic polynomial approximate value of $y(3)$. [8]

(d) Explain very briefly why the quadratic polynomial found in part (c) may not give a good approximation to the value of $y(7)$. [2]

Question 5 (*Unit 10*) – 14 marks

Do *not* use your computer to answer this question. Your solution must show your working.

(a) Consider the matrix

$$\mathbf{B} = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

By hand, solve the characteristic equation for $\mathbf{B}$ and hence show that the eigenvalues are 0, 2 and 4. For each eigenvalue, find a corresponding eigenvector. [10]

(b) Find the eigenvalues of $\mathbf{B}^2 + \mathbf{I}$ and $(\mathbf{B}^2 + 3\mathbf{B} - \mathbf{I})^{-1}$, justifying your answers. What is the eigenvector corresponding to the eigenvalue of largest magnitude for the matrix $\mathbf{B}^2 + \mathbf{I}$? [4]

Question 6 (*Unit 10*) – 11 marks

You may use your computer in any part of this question to multiply matrices and vectors; if you do, you must include all your Mathcad output. You should not use the worksheet `20910-01 Eigenvalues and eigenvectors.xmcd`.

All final answers should be given to three decimal places.

This question is concerned with the numerical calculation of the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -2 \\ 7 & 3 & -8 \end{bmatrix}.$$

(a) Take $\mathbf{e}_0 = [1 \quad 0 \quad 0]^T$ and use direct iteration (Procedure 4.1 on page 84 of *Unit 10*) to calculate $\mathbf{e}_1$ and $\mathbf{e}_2$. Given that $\mathbf{e}_8 = [0.2783 \quad 0.2265 \quad 1.0000]^T$, where the values are correct to four decimal places, calculate $\mathbf{e}_9$ correct to four decimal places. What can you conclude about the eigenvalues and eigenvectors of $\mathbf{A}$? If you can estimate the values, then do so to three decimal places. *[4]*

(b) Now consider the inverse iteration method (Procedure 4.2 on page 85 of *Unit 10*) applied to the matrix $\mathbf{A}$. The inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 2 & 0 \\ 2 & 6 & -2 \\ -1 & 4 & -1 \end{bmatrix}.$$

Starting with $\mathbf{e}_0 = [0 \quad 1 \quad 0]^T$, the inverse iteration method gives $\mathbf{e}_{20} = [0.2713 \quad 1.0000 \quad 0.5852]^T$, correct to four decimal places. Calculate $\mathbf{e}_{21}$ correct to four decimal places. What can you conclude about the eigenvalues and eigenvectors of $\mathbf{A}$? If you can estimate the values, then do so to three decimal places. *[3]*

(c) Calculate the trace of the matrix $\mathbf{A}$, and hence find the third eigenvalue of $\mathbf{A}$. *[2]*

(d) Check your answers by using Mathcad to determine the eigenvalues and eigenvectors of $\mathbf{A}$ using `eigenvals(A)=` and `eigenvecs(A)=`, and briefly comment on any discrepancies. *[2]*

PC

Question 7 (*Unit 11*) – 20 marks

Do *not* use your computer in this question; your solutions must show your working.

(a) (i) Express the following inhomogeneous system of first-order differential equations for $x(t)$ and $y(t)$ in matrix form:

$$\dot{x} = -2x - y + 12t + 12,$$
$$\dot{y} = 2x - 5y - 5.$$

[1]

(ii) Write down, also in matrix form, the corresponding homogeneous system of differential equations. *[1]*

(iii) Find the eigenvalues of the matrix of coefficients and an eigenvector corresponding to each eigenvalue. Hence write down the complementary function for the system of differential equations. *[5]*

(iv) Calculate a particular integral for the inhomogeneous system, and hence write down the general solution. *[4]*

(v) Determine the particular solution of the initial-value problem with the initial conditions $x(0) = 3$ and $y(0) = 2$. *[4]*

(b) An object moves in the plane in such a way that its Cartesian coordinates (x, y) at time t satisfy the following homogeneous system of second-order differential equations:

$$\ddot{x} = -2x - y,$$
$$\ddot{y} = 2x - 5y.$$

Express the system in matrix form. Write down the general solution of the system. Explain briefly how the system may undergo simple harmonic motion in a straight line in two distinct ways. For each such simple harmonic motion, determine the angular frequency and a vector giving the direction of motion. [*5*]

Question 8 (*Unit 11*) – 5 marks

This question uses computer algebra to find the general solution of the following system of first-order differential equations:

$$\dot{x} = 3x + 2y - 2z,$$
$$\dot{y} = 2x + 3y + 4z,$$
$$\dot{z} = 2x + 4y + 3z.$$

(a) Write the system of equations in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 3×3 matrix. [*1*]

(b) Use Mathcad to determine the eigenvalues of $\mathbf{A}$ and corresponding eigenvectors using `eigenvals(A)=` and `eigenvecs(A)=`. Include a printout of your results. Do not use the worksheet `20911-02 1st-order system of 3 Diff Eqns.xmcd`. [*1*] PC

(c) Hence give the general solution of the above system of differential equations in vector form. What is the long-term behaviour of the solution? [*3*]

This assignment covers *Units 1–11*.

Question 1 (*Unit 1*)

Consider the real function f defined by the formula
$f(x) = \ln((x+2)(5-x))$. Which option gives the largest possible domain for f?

Options

A $x > -2$ **B** $-2 < x < 5$ **C** $x < -2$

D $x > 5$ **E** $x < 5$ **F** $x > 0$

G $0 < x < 5$ **H** $x < -2$ and $x > 5$

Question 2 (*Unit 2*)

Consider the differential equation

$$\frac{dy}{dx} = x^2 + x^2 y.$$

Which of the following options is correct?

Options

A The differential equation may be solved using the separation of variables method but not the integrating factor method.

B The differential equation may be solved using the integrating factor method but not the separation of variables method.

C The differential equation may be solved using either the integrating factor method or the separation of variables method.

D The differential equation cannot be solved using either the integrating factor method or the separation of variables method.

Question 3 (*Unit 3*)

Select the option that gives an expression for the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

Options

A $y = Ae^{2x} + Be^{3x}$ **B** $y = e^{-5x/2}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

C $y = Ae^{2x} + Be^{-3x}$ **D** $y = e^{-x/2}(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x)$

E $y = Ae^{-2x} + Be^{3x}$ **F** $y = e^{5x/2}(A\cos\frac{1}{2}x + B\sin\frac{1}{2}x)$

G $y = Ae^{-2x} + Be^{-3x}$ **H** $y = e^{x/2}(A\cos\frac{3}{2}x + B\sin\frac{3}{2}x)$

Question 4 (*Unit 4*)

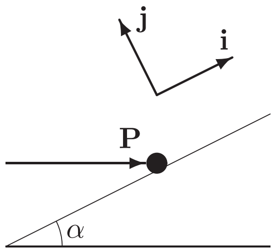
The vector perpendicular to $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where O, A and B are points with coordinates $(0,0,0)$, $(3,6,1)$ and $(1,2,3)$, respectively, is of the form $x\mathbf{i}+y\mathbf{j}$. Which option gives the values of x and y?

Options

A	$x=-16$, $y=-8$	**B**	$x=-16$, $y=8$
C	$x=-8$, $y=16$	**D**	$x=16$, $y=8$
E	$x=-20$, $y=-10$	**F**	$x=10$, $y=-20$
G	$x=-10$, $y=20$	**H**	$x=20$, $y=10$

Question 5 (*Unit 5*)

A horizontal force $\mathbf{P}$ acts on a particle lying on a rough plane inclined at an angle α to the horizontal. The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel and perpendicular to the plane, as shown in the diagram below.



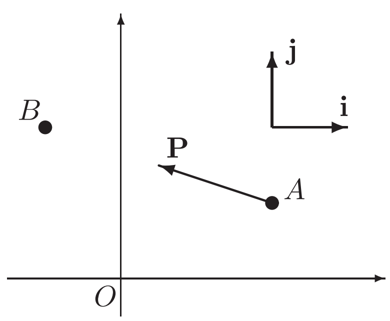
Which option gives the correct expression for $\mathbf{P}$ in terms of $\mathbf{i}$ and $\mathbf{j}$?

Options

A	$\lvert\mathbf{P}\rvert(\sin(\alpha)\,\mathbf{i}+\cos(\alpha)\,\mathbf{j})$	**B**	$\lvert\mathbf{P}\rvert(-\sin(\alpha)\,\mathbf{i}+\cos(\alpha)\,\mathbf{j})$
C	$\lvert\mathbf{P}\rvert(-\sin(\alpha)\,\mathbf{i}-\cos(\alpha)\,\mathbf{j})$	**D**	$\lvert\mathbf{P}\rvert(\sin(\alpha)\,\mathbf{i}-\cos(\alpha)\,\mathbf{j})$
E	$\lvert\mathbf{P}\rvert(\cos(\alpha)\,\mathbf{i}+\sin(\alpha)\,\mathbf{j})$	**F**	$\lvert\mathbf{P}\rvert(-\cos(\alpha)\,\mathbf{i}+\sin(\alpha)\,\mathbf{j})$
G	$\lvert\mathbf{P}\rvert(-\cos(\alpha)\,\mathbf{i}-\sin(\alpha)\,\mathbf{j})$	**H**	$\lvert\mathbf{P}\rvert(\cos(\alpha)\,\mathbf{i}-\sin(\alpha)\,\mathbf{j})$

Question 6 (*Unit 5*)

A force $\mathbf{P}$ of magnitude $4\,\mathrm{N}$ is directed from a point A towards a point B. The position vectors of points A and B relative to O in the coordinate system shown below are $2\mathbf{i}+\mathbf{j}$ and $-\mathbf{i}+2\mathbf{j}$, respectively.



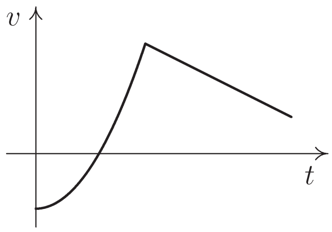
Select the option that gives the torque of $\mathbf{P}$ about O.

Options

A	$20\,\mathbf{k}$	**B**	$-20\,\mathbf{k}$	**C**	$2\sqrt{10}\,\mathbf{k}$	**D**	$-2\sqrt{10}\,\mathbf{k}$
E	20	**F**	-20	**G**	$2\sqrt{10}$	**H**	$-2\sqrt{10}$

Question 7 (*Unit 6*)

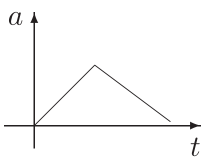
A particle is moving along a straight line with velocity $v(t)\,\mathbf{i}$. The graph of $v(t)$ against t is shown below.



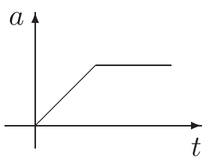
At any instant, the acceleration of the car is $a(t)\,\mathbf{i}$. Which option gives a possible graph of $a(t)$ against t?

Options

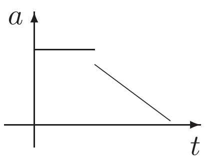
A



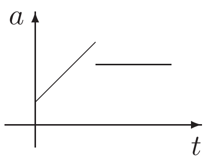
B



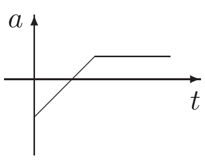
C



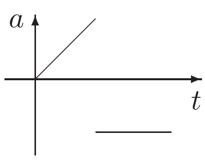
D



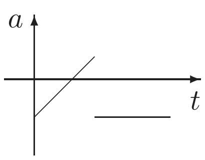
E



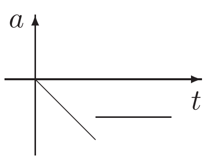
F



G

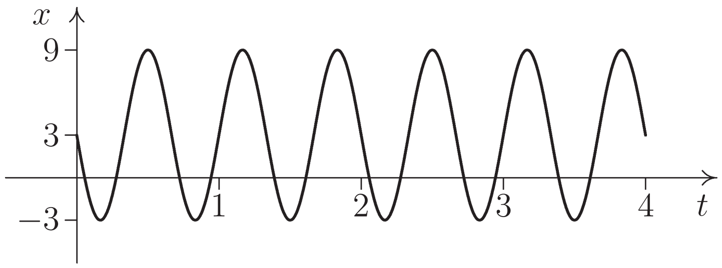


H



Question 8 (*Unit 7*)

The position–time graph of a particle oscillating horizontally between two springs is shown below.



Which option gives the most appropriate equation for the displacement as a function of time?

Options

A $x = 3 - 6\sin(2\pi t)$ **B** $x = 3 - 3\sin(2\pi t)$

C $x = 3 - 6\sin\left(\frac{3}{2}\pi t\right)$ **D** $x = 3 - 6\sin(3\pi t)$

E $x = 3 - 3\sin\left(\frac{3}{2}\pi t\right)$ **F** $x = 3 + 6\sin(3\pi t)$

G $x = 3 - 3\cos\left(\frac{3}{2}\pi t\right)$ **H** $x = 3 - 6\cos(3\pi t)$

Question 9 (*Unit 8*)

A particle of mass m moves under the influence of a force $\mathbf{F}(x) = (x + \sin x)\mathbf{i}$ along the $\mathbf{i}$-axis. Its displacement from a fixed point is x. Select the option that could represent the total mechanical energy of the particle at any instant.

Options

A $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}x^2 - \sin x$ **B** $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}x^2 + \sin x$

C $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2 + \sin x$ **D** $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2 - \sin x$

E $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}x^2 + \cos x$ **F** $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2 + \cos x$

G $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2 - \cos x$ **H** $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}x^2 - \cos x$

Question 10 (*Unit 9*)

M is the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are real numbers. Consider the following statements.

P: If $b = c$, the eigenvalues are real,
Q: If $bc = 0$, the eigenvalues are a and d.
R: If the eigenvalues are equal, then they are real.

Which option is correct?

Options

A P, Q and R are true
B Only P and Q are true
C Only P and R are true
D Only Q and R are true
E Only P is true
F Only Q is true
G Only R is true
H None of P, Q and R is true

Question 11 (*Unit 10*)

The vector $[-1 \quad 1 \quad 0]^T$ is an eigenvector of the matrix

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Select the option that gives the corresponding eigenvalue.

Options

A 0 **B** 1 **C** -1 **D** 2
E -2 **F** 4 **G** -4 **H** None of these options

Question 12 (*Unit 11*)

The complementary function of the system of differential equations

$$\dot{x} = 11x - 18y + 9e^t,$$
$$\dot{y} = 6x - 10y,$$

is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + \beta \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-t}.$$

Select the option that gives a suitable candidate for a particular integral for this system.

Options

A $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pe^t \\ 0 \end{bmatrix}$
B $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pe^t \\ qe^t \end{bmatrix}$
C $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pte^t \\ 0 \end{bmatrix}$
D $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pte^t \\ qte^t \end{bmatrix}$
E $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pte^t \\ qe^t \end{bmatrix}$
F $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pte^t \\ pte^t \end{bmatrix}$
G $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pe^t \\ pe^t \end{bmatrix}$
H $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} pe^t \\ qte^t \end{bmatrix}$
