M343

Assignment Booklet 2013J

Contents **Cut-off date**

Please send all your answers to each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the appropriate cut-off date shown above.

Your tutor will inform you about the address to use for submitting your TMAs. Please don't submit your TMAs directly to the University. Regrettably, the University is unable to accept TMAs submitted electronically on M343. If you have any questions about how best to prepare and submit your TMAs, please contact your tutor.

You will find instructions on how to fill in the PT3 form in the online *Assessment Handbook* for undergraduate modules. Remember to fill in the correct assignment number as listed above, and allow sufficient time in the post for each assignment to reach its destination on or before the cut-off date. Do not use recorded delivery.

The marks allocated to each part of each TMA question are indicated in brackets in the margin.

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3.1

Please show your working for all questions. This will give your tutor the opportunity to award you some marks for a question where your working is partially correct even though you may not have the correct final answer.

You are advised to keep copies of your assignments in case of loss in the mail.

Computer-marked assignments

CMAs must be submitted electronically via the eCMA system. Instructions for using the eCMA system are given below.

Please ensure that you submit each eCMA well in advance and no later than **midday on the cut-off date (UK local time)**. eCMAs received after that time will be marked 'L' (late) and given a zero score. You are advised not to wait until the deadline in case of last minute computer or internet problems.

Please note that for CMAs your score and a list of correct responses will be available *only* via StudentHome. They will not be sent to you in the post.

Instructions for using the eCMA system

Go to your StudentHome page and find the link to 'M343 Module Record'. The system is straightforward to use, but it would be a good idea to make sure that you know how to access and use it well before the cut-off date. Once you have entered the eCMA system, please ensure that you click on the correct module code and assignment number.

You can enter a few answers at a time and come back and change them as often as you like, but once you have *submitted* your answers you cannot change them. It is therefore essential that you carefully check that everything is as you intended, and ensure that you have made some response for every question, even if it is 'don't know', before you submit your answers. Once you have submitted your eCMA, you will receive an on-screen notification that it has been received by the eCMA system.

Questions 1 to 8 below, on *Book 1*, and Parts I and II of *Book 2*, form tutor-marked assignment M343 01. The marks available for the questions are indicated at the beginning of each question.

Please send all your answers to your tutor, along with an appropriately completed assignment form (PT3). Be sure to fill in the assignment number on this form as

$\boxed{\text{M343 01}}$.

Question 1 – 8 marks

This question is intended to assess your understanding of the rules of probability, the Theorem of Total Probability and Bayes' formula.

You should be able to answer this question after working through Section 1 of Book 1.

A factory has two production lines for kettles, 1 and 2 (say). Line 1 produces four times as many kettles as line 2. The probability that a kettle from line 1 will prove to be faulty when checked is 0.015. The probability that a kettle from line 2 will prove to be faulty is 0.025.

(a) Identify the events involved by giving them names, and write down the information supplied as probabilities using these names. (For example: let E_1 be the event that a kettle was produced on line 1; $P(E_1) = \ldots .$) [*2*]

(b) Calculate the probability that a randomly selected kettle will prove to be faulty. [*1*]

(c) Given that a kettle is found to be faulty, calculate the probability that it was produced on line 1. [*2*]

(d) Given that a kettle is not faulty, calculate the probability that it was produced on line 2. [*3*]

Be sure to state clearly the results from *Book 1* that you use.

Question 2 – 11 marks

This question is intended to assess your understanding of discrete random variables, expectation for discrete random variables, and simulation from a discrete distribution.

You should be able to answer this question after working through Section 2 and Subsection 6.2 of Book 1.

There are five basket-only checkouts in the food hall of my local department store. Observation over many months suggests that the number of basket-only checkouts open at any one time may be reasonably modelled as a discrete random variable X with the probability distribution given below.

x	2	3	4	5
$P(X = x)$	0.05	0.15	0.35	0.45

(a) Calculate the value of the c.d.f. $F(x) = P(X \leq x)$ at each of the points $x = 0, 1, 2, 3, 4, 5$, and display these values in a table. [2]

(b) Calculate the mean and variance of the number of checkouts open at any time. (Use Formula (2.3) of *Book 1* to find the mean in this part of the question.) [3]

(c) Use Formula (2.4) of *Book 1* to calculate the mean number of checkouts open at any time. [2]

(d) Simulate the number of checkouts open on each of two occasions that I go to the food hall. (That is, simulate two observations x_1, x_2 from the distribution of X.) Use the two numbers $u_1 = 0.114\,57$, $u_2 = 0.437\,64$, which are random observations from the uniform distribution $U(0, 1)$. (They were obtained by taking groups of five random digits from the fifteenth row of Table 5 in the *Handbook*.) [2]

(e) (i) Describe an alternative procedure for simulating observations of the number of open checkouts, that uses pairs of digits from Table 5 in the *Handbook* (instead of random observations from $U(0, 1)$).

(ii) Illustrate your procedure by simulating five observations using the pairs 42 15 64 76 31 from the fifteenth row of the table. [2]

Question 3 – 10 marks

This question is intended to assess your understanding of probability generating functions.

You should be able to answer this question after working through Sections 2 and 3 of Book 1.

(a) The random variables V and W have the following probability generating functions.

(i) $\Pi_V(s) = \dfrac{7s}{12 - 5s}$ (ii) $\Pi_W(s) = \left(\dfrac{5s + 7}{12}\right)^6$

Use the table of discrete probability distributions in the *Handbook* to identify the distribution of each of the random variables. In each case, you should name a family of distributions and give the values of any parameters. [2]

(b) The number of pairs of shoes bought by each customer entering a shoe shop is a random variable X that has a geometric distribution starting at 0 with mean $\frac{3}{4}$.

(i) Find the value of the parameter p of the geometric distribution, and hence write down the probability generating function of X. [*3*]

(ii) Eight customers visit the shoe shop. The number of pairs of shoes that a customer buys is independent of the number bought by any other customer. Write down the probability generating function of Y, the total number of pairs of shoes that the eight customers buy. Use the table of discrete probability distributions in the *Handbook* to identify the distribution of Y. Hence find the mean and variance of the total number of pairs of shoes purchased by the eight customers. [*5*]

Question 4 – 15 marks

This question is intended to assess your understanding of continuous random variables, expectation for continuous random variables and simulation from a continuous distribution.

You should be able to answer this question after working through Section 4 and Subsection 6.1 of Book 1.

At my local corner shop, I have noticed that it always takes at least two minutes to serve a customer, and it can take much longer. The time in minutes that it takes to serve a customer may be modelled by a continuous random variable T with probability density function

$$f(t) = \frac{24}{t^4}, \quad t \geq 2.$$

(a) Show that the c.d.f. of the random variable T is given by

$$F(t) = 1 - \frac{8}{t^3}, \quad t \geq 2.$$ [*2*]

(b) According to the model, what proportion of customers take more than five minutes to serve? [*2*]

(c) Calculate the probability that it takes between four and eight minutes to serve a customer. [*2*]

(d) Use the p.d.f. $f(t)$ to calculate the mean and variance of the time taken to serve a customer. [*5*]

(e) Use Formula (4.4) of *Book 1* to calculate the mean time taken to serve a customer. [*2*]

(f) Simulate the time taken to serve a customer, using the number $u = 0.3169$, which is a random observation from the uniform distribution $U(0, 1)$. Give your answer in minutes and seconds correct to the nearest second. [*2*]

Question 5 – 13 marks

This question is intended to assess your understanding of the properties of the exponential distribution, and of the notion of a random process and the notation used for random processes.

You should be able to answer this question after working through Section 5 of Book 1 and Part I of Book 2.

In the food hall of my local department store, there is a central queue for the five basket-only checkouts. From past experience, I know that the service times of three of the five assistants are exponentially distributed with mean 3 minutes, and those of the other two assistants are exponentially distributed with mean 4 minutes.

One day Christine joins the central queue when all five checkouts are open and all five assistants are busy, but no one is waiting to be served.

(a) Find the distribution of the time that Christine will have to wait before she can move forward for service. Show that her expected waiting time is 40 seconds. *[3]*

(b) Calculate the probability that Christine will have to wait for more than two minutes until an assistant is free to serve her. *[2]*

(c) Three minutes after joining the queue, Christine is still waiting. Another customer, Derek, stands behind her in the queue. (So now the same five people are being served as at the start, and two people are standing in the queue.)

State the distribution of Derek's waiting time before his service commences. Give a brief explanation for your answer.

Hence calculate the mean and standard deviation of Derek's waiting time. *[4]*

(d) Suggest two random processes associated with the queueing system, one with a discrete time domain and one with a continuous time domain. In your examples, you should use the notation for random processes introduced in Part I of *Book 2*; for example, $\{X(t); t \geq 0\}$ or $\{X_n; n = 0, 1, \ldots\}$. Describe precisely what the random variables ($X(t)$, X_n or whatever) represent, and state clearly the state space corresponding to each process. (See *Book 2*, Activity 2.5 for the sort of thing that is required here.) *[4]*

Question 6 – 20 marks

This question is intended to assess your understanding of the Poisson process and the multivariate Poisson process, and your use of the standard notation for some of the random variables associated with these processes.

You should be able to answer this question after working through Sections 3 and 5 of Book 2.

In this question, you should use the standard notation introduced in *Book 2* for the random variable representing the number of events that occur in an interval of fixed length in a Poisson process. Credit will be given for using this notation.

On Friday mornings, customers join the queue at a supermarket's basket-only checkout according to a Poisson process with rate 30 per hour.

(a) Calculate the probability that during a five-minute period:

(i) no customer joins the queue;

(ii) at least three customers join the queue. *[5]*

(b) Use the random numbers from exponential distributions in Table 6 of the *Handbook* to simulate the times, to the nearest second, at which the first three customers join the queue after 10 o'clock one Friday morning. Use the fifteenth row of the table (beginning $3.2409, 0.1969, \ldots$). *[3]*

Customers may be classified into three groups according to their method of payment. Experience has shown that 60% will pay cash (group A), 10% will pay using the supermarket's store card (group B), and the rest will pay using a credit card (group C).

(c) State the distribution of the number of customers who pay using a credit card in a ten-minute period. Calculate the probability that fewer than two customers who pay using a credit card join the queue in a ten-minute period. *[4]*

(d) Calculate the probability that exactly six customers join the queue in a fifteen-minute period, exactly four of whom pay cash. *[6]*

(e) Seven customers join the queue between 10 am and 10:20 am one Friday. Calculate the probability that exactly five of them pay cash. *[2]*

Question 7 – 13 marks

This question is intended to assess your understanding of the non-homogeneous Poisson process, and your use of the standard notation for some of the random variables associated with a point process.

You should be able to answer this question after working through Section 6 of Book 2.

In this question, you should use the standard notation for the random variables in a non-homogeneous Poisson process (as in Section 6 of *Book 2*). Credit will be given for using this notation.

At a large hotel, guests sometimes arrive seeking accommodation without having a reservation. The incidence of such casual room bookings on a Friday afternoon and evening may be modelled as a non-homogeneous Poisson process with hourly rate

$$\lambda(t) = \frac{4}{5+4t}, \quad 0 \leq t \leq 12,$$

where t is the elapsed time in hours since midday. No casual booking is accepted after midnight.

(a) Show that the expected number of casual bookings made by t hours after midday is $\mu(t) = \log(1 + 0.8t)$. [*2*]

(b) Calculate the probability that exactly three casual bookings are made between midday and 5 pm. [*2*]

(c) Calculate the probability that fewer than two casual bookings are made after 10 pm. [*3*]

(d) For $j = 1, 2, \ldots$, given that the time at which the jth casual booking is made is w_j, show that w_{j+1}, the simulated time at which the $(j+1)$th booking is made, may be obtained using the recurrence relation

$$w_{j+1} = \frac{4w_j + 5u}{4(1-u)},$$

where u is a random observation from the uniform distribution $U(0,1)$. [*3*]

(e) Use the formula in part (d) to simulate the times of arrival of guests making casual bookings between midday and midnight on a Friday, using groups of five digits from the nineteenth row of Table 5 in the *Handbook* (beginning 43333, 63289, . . .). State clearly the times on the clock at which the bookings are made, and give the times to the nearest minute. [*3*]

Question 8 – 10 marks

This question is intended to assess your understanding of the compound Poisson process and the index of dispersion for a random process.

You should be able to answer this question after working through Sections 7 and 8 of Book 2.

Loads of freight arrive at a dockside cargo office according to a Poisson process with rate 24 per hour. Each load contains one or more packages: Y, the number of packages in a load, has the probability distribution below. (The number of packages in a load is independent of the number in any other load.)

y	1	2	3	4	5
$P(Y = y)$	0.05	0.05	0.1	0.45	0.35

(a) Calculate the mean and variance of the number of packages in a load. [*3*]

(b) Calculate the mean and variance of the total number of packages that arrive at the cargo office in an afternoon (between 1 pm and 6 pm). [*3*]

(c) Calculate the index of dispersion for this process. What does your value of the index of dispersion tell you about the pattern of arrivals of packages at the cargo office? [*4*]

Note that sometimes there will be more than one correct option for a particular question. Where this is so (and you will be told when this is the case), you should identify all the correct options.

Questions 1 to 7 are on Part III of Book 2.

Questions 1 to 3

In a large meadow, the positions of wild flowers of a particular species may be assumed to be randomly located according to a two-dimensional Poisson process with density $\lambda = 0.4$ per square metre.

1 Choose the option that gives the expected distance (in metres, correct to three significant figures) from a randomly selected point in the meadow to the nearest flower of this species.

Options for Question 1

A 1.25 **B** 2.50 **C** 0.125 **D** 0.791 **E** 7.91

2 Choose the option that gives the probability (correct to three decimal places) that the distance from the randomly selected point to the nearest flower is less than 60 centimetres.

Options for Question 2

A 0.213 **B** 0.364 **C** 0.636 **D** 0.787 **E** 0.959

3 Choose the option that gives the probability (rounded to three decimal places) that the distance from a randomly selected flower to the flower nearest to it is more than 1 metre.

Options for Question 3

A 0.007 **B** 0.285 **C** 0.330 **D** 0.670 **E** 0.715

Questions 4 to 7

In an investigation into the disposition of a certain species of small shrub in a wood, ten R-distances (point-to-nearest-object distances) and ten S-distances (object-to-nearest-object distances) are calculated. These are shown below.

R-distances	12.5	8.1	15.2	13.4	17.9	15.3	14.2	9.3	11.7	8.8
S-distances	4.2	11.4	6.1	6.9	8.3	10.2	4.4	5.6	3.3	10.9

Hopkins' test is used. A fixed-level test with a 5% significance level is carried out.

4 Choose the option that gives the value of the test statistic (correct to three decimal places).

Options for Question 4

A 0.564 **B** 3.143 **C** 1.773 **D** 0.347 **E** 2.884

5 Choose the option that gives the rejection region for the test.

Options for Question 5

A $0.269 \leq h \leq 3.717$ **B** $h \leq 0.269$ and $h \geq 3.717$

C $0.406 \leq h \leq 2.464$ **D** $h \leq 0.406$ and $h \geq 2.464$

E $0.471 \leq h \leq 2.124$ **F** $h \leq 0.471$ and $h \geq 2.124$

6 Choose the option that gives a valid conclusion of the test.

Options for Question 6

A The hypothesis that the shrubs are randomly located in the wood can be rejected at the 5% significance level.

B The hypothesis that the shrubs are not randomly located in the wood can be rejected at the 5% significance level.

C There is insufficient evidence at the 5% significance level to reject the hypothesis that the shrubs are not randomly located in the wood.

D There is insufficient evidence at the 5% significance level to reject the hypothesis that the shrubs are randomly located in the wood.

7 Choose the option that gives a correct statement.

Options for Question 7

A Since the observed value of the test statistic is low, the data suggest that the shrubs are somewhat regularly located.

B Since the observed value of the test statistic is low, the data suggest that there is clustering of the shrubs.

C Since the observed value of the test statistic is high, the data suggest that the shrubs are somewhat regularly located.

D Since the observed value of the test statistic is high, the data suggest that there is clustering of the shrubs.

Questions 8 and 9

The offspring random variable X of a Galton–Watson branching process that starts with a single ancestor in generation 0 has p.g.f.

$$\Pi(s) = \frac{1+s}{7-5s}.$$

8 Choose TWO options, one that gives the mean of X and one that gives the variance of X.

Options for Question 8

A 1 **B** 3 **C** 6 **D** 9 **E** 12 **F** 15

9 Choose TWO options, one that gives the mean and one that gives the variance of the number of individuals in the third generation.

Options for Question 9

A 9 **B** 27 **C** 729 **D** 1053 **E** 9477

Question 10

Each of the probability distributions listed in the options below is the offspring distribution for a Galton–Watson branching process. Choose the THREE options that give distributions for which eventual extinction of the branching process is possible but not certain.

Options

A Poisson(1.1)

B $G_1(0.8)$

C $G_0(0.4)$

D $B(10, 0.1)$

E Negative binomial with range $\{0, 1, \ldots\}$ and parameters $r = 3$ and $p = 0.4$

F Modified geometric with parameters $a = 5$, $b = 2$, $c = 10$ and $d = 7$

Question 11

The offspring random variable of a Galton–Watson branching process has the negative binomial distribution with range $\{0, 1, \ldots\}$ and parameters $r = 6$ and $p = \frac{1}{3}$. Choose the option that gives the probability of eventual extinction of the branching process correct to three decimal places.

Options

A 0.108 **B** 0.109 **C** 0.110 **D** 0.111

Questions 12 to 15

The value of a stock market share may be assumed to vary from day to day according to the following rule: if it is worth £n one day ($n > 0$), then with probability p it will be worth £$(n+1)$ the next day and with probability $q(= 1 - p)$ it will be worth £$(n - 1)$.

Gary owns one share. Today it is worth £8. Gary has decided to sell the share immediately if its value either drops to £5 (in which case he will have made a loss) or rises to £15 (in which case he will have made a profit).

12 Choose the option that gives the expected number of days (rounded to the nearest day) until Gary sells the share when $p = \frac{2}{3}$.

13 Choose the option that gives the expected number of days (rounded to the nearest day) until Gary sells the share when $p = 0.5$.

Options for Questions 12 and 13

A 21 **B** 56 **C** 35 **D** 17 **E** 69

14 Choose the option that gives the probability (correct to four decimal places) that Gary will make a profit when he sells his share when $p = \frac{2}{3}$.

15 Choose the option that gives the probability (correct to four decimal places) that Gary will make a loss when he sells his share when $p = 0.5$.

Options for Questions 14 and 15

A 0.0039	**B** 0.1241	**C** 0.3000	**D** 0.4667
E 0.5333	**F** 0.7000	**G** 0.8759	**H** 0.9961

Questions 16 to 20

A particle executes a simple unrestricted random walk on the line, a step to the right of length 1 occurring with probability $\frac{4}{7}$, and a step to the left of length 1 occurring with probability $\frac{3}{7}$. Initially, the particle is at the origin.

16 Choose the option that gives the probability (correct to four decimal places) that the particle returns to the origin for the first time after 10 steps.

17 Choose the option that gives the probability that the particle ever returns to the origin.

18 After some time, the particle is observed to be located at $+10$ (ten steps to the right of the origin). Choose the option that gives the probability that the particle will ever return to the origin from $+10$.

Options for Questions 16 to 18

A	0.0075	**B**	0.0247	**C**	0.0563
D	0.2220	**E**	0.8571	**F**	1.0000

19 Choose TWO options, one that gives the mean and one that gives the variance of X_{98}, the position of the particle after 98 steps.

Options for Question 19

A	-14	**B**	14	**C**	24	**D**	56	**E**	96	**F**	98

20 Choose the option that gives an approximate value for the probability that the particle will be less than 10 units from the origin after 98 steps, calculated using an appropriate continuity correction.

Options for Question 20

A	0.075	**B**	0.296	**C**	0.315	**D**	0.334
E	0.666	**F**	0.704	**G**	0.685	**H**	0.925

Questions 1 to 6 below, on Part III of *Book 2*, *Book 3*, and Part I of *Book 4*, form tutor-marked assignment M343 02. The marks available for the questions are indicated at the beginning of each question.

Please send all your answers to your tutor, along with an appropriately completed assignment form (PT3). Be sure to fill in the assignment number on this form as

$\boxed{\text{M343 02}}$.

Question 1 – 18 marks

This question is intended to assess your understanding of the two-dimensional Poisson process and the χ^2 dispersion test.

You should be able to answer this question after working through Part III of Book 2.

(a) In a particular wood, diseased trees may be assumed to be randomly located according to a two-dimensional Poisson process with density 16 per square kilometre. A rectangular region of the wood 500 metres long by 400 metres wide is marked out for detailed study.

(i) Write down the probability distribution of N, the number of diseased trees in the region. [*1*]

(ii) Calculate the probability that the region contains more than two diseased trees. [*3*]

(iii) Simulate the number of diseased trees in this region using the number $u = 0.1784$, which is a random observation from the uniform distribution $U(0, 1)$. [*3*]

(iv) (1) Explain briefly (in one or two sentences) how you would simulate the positions of the diseased trees in this region.

(2) Use the method that you have just described and groups of five digits from the second row of the table of random digits in the *Handbook* (beginning 95448, 70838, ...) to simulate the positions of the diseased trees in the region. Give the coordinates of the diseased trees to the nearest metre.

(3) Plot the positions of the diseased trees on a sketch of the region. (*No graph paper or separate pieces of paper are required for this.*) [*4*]

(b) A wood is split into 21 equal-sized quadrats and the number of trees of a particular species in each quadrat is counted. The results are as follows.

5	7	5	6	7	5	5
4	9	8	5	7	4	5
3	4	7	8	6	4	5

Check that it is appropriate to use the χ^2 dispersion test to investigate whether the trees could reasonably be supposed to be randomly positioned in the wood, and carry out the test. You may use either a fixed-level test or a significance test, whichever you prefer.

- If you decide to carry out a fixed-level test, then you should use a 5% significance level and state clearly the rejection region for your test.
- If you decide to carry out a significance test, then you should give an interval within which the p value for the test lies.

If your analysis suggests that the trees are not randomly located, then say how you think they are located, justifying your answer briefly. [*7*]

Question 2 – 12 marks

This question is intended to assess your understanding of the Galton–Watson branching process.

You should be able to answer this question after working through Part I of Book 3.

(a) Suggest a random phenomenon, different from any mentioned in Part I of *Book 3*, for which the Galton–Watson branching process might provide a reasonable model. Be clear about the random variable being counted (Z_n) and the interpretation of the notion of a 'generation'. [*3*]

(b) The offspring random variable of a Galton–Watson branching process with a single ancestor in generation 0 has the negative binomial distribution with range $\{0, 1, \ldots\}$ and parameters $r = 2$ and $p = 0.4$.

(i) Write down the p.g.f. of the offspring random variable. [*1*]

(ii) Calculate each of the following probabilities correct to four decimal places.

(1) The probability of extinction by the second generation.

(2) The probability of extinction at the fourth generation. [*4*]

(iii) Find the probability that the process will eventually become extinct. *Do not use an iterative method to obtain this probability.* [*4*]

Question 3 – 18 marks

This question is intended to assess your understanding of random walks on the line.

You should be able to answer this question after working through Part II of Book 3.

(a) Suggest an example, different from any mentioned in Part II or in Section 15 of *Book 3*, of a situation that could reasonably be modelled by a particle executing a simple random walk on the line with two reflecting barriers. Be sure to explain why a simple random walk might be a suitable model and what reflection at the barriers represents. *[3]*

(b) A particle executes an unrestricted random walk on the line starting at the origin. The ith step, Z_i, has the following distribution:

$$P(Z_i = 1) = p, \quad P(Z_i = -4) = q = 1 - p.$$

(i) Find the mean and variance of Z_i, and hence find the mean and variance of X_n, the position of the particle after n steps. *[4]*

(ii) Find the probability distribution of X_n, the position of the particle after n steps, and state the range of X_n. Explain all the steps in your derivation of the distribution. *[5]*

(iii) Calculate the following probabilities in terms of p and q:

$$P(X_8 = -2), \quad P(X_9 = 1).$$ *[2]*

(iv) Calculate the probabilities u_0, u_5, u_{10} in terms of p and q, and hence find the probability that the particle returns to the origin for the first time after 10 steps. *[4]*

Question 4 – 22 marks

This question is intended to assess your understanding of Markov chains.

You should be able to answer this question after working through Part III of Book 3.

(a) A Markov chain has the transition matrix

$$\mathbf{P} = \begin{array}{c} \mathit{1} \\ \mathit{2} \\ \mathit{3} \\ \mathit{4} \\ \mathit{5} \\ \mathit{6} \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0.7 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0.6 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 \end{pmatrix}.$$

(i) Draw a diagram to represent the Markov chain with transition matrix $\mathbf{P}$. Identify the communicating classes, and state whether each class is closed or not closed. *[3]*

(ii) Say whether or not this Markov chain has a unique stationary distribution, justifying your answer. *[1]*

(b) Every Thursday evening, when the Dolans go to the supermarket to buy their groceries for the following week, they buy a large pot of yoghurt. They always choose natural (N), apricot (A), blueberry (B) or strawberry (S). If they buy natural yoghurt one week, then they will buy one of the three fruit-flavoured varieties the following week with equal probability. After buying apricot, they buy either blueberry or strawberry the next week, and they are twice as likely to buy strawberry as they are to buy blueberry. After buying blueberry, they are equally likely to buy either blueberry or strawberry the next week. After buying strawberry, they buy either natural or apricot the next week, and they are twice as likely to buy natural as they are to buy apricot.

(i) Write down the matrix of transition probabilities for the Markov chain for weekly yoghurt purchases described above. [*2*]

(ii) One week the Dolans buy strawberry yoghurt.

(1) What is the probability that they will buy natural yoghurt the next week?

(2) Calculate the probability that they will buy blueberry yoghurt in two weeks' time.

(3) Calculate the probability that they will buy natural yoghurt the next week *and* blueberry yoghurt in two weeks' time. [*3*]

(iii) In the long run, in what proportion of weeks do the Dolans buy each of the four varieties of yoghurt? Show all the steps in your working, and give your answers either as fractions or as decimals correct to three decimal places. [*6*]

(iv) If the Dolans buy apricot yoghurt one week, what is the expected number of weeks until they next buy apricot yoghurt? [*1*]

Immediately after returning from a holiday, the Dolans always buy either natural yoghurt or apricot yoghurt, and they are twice as likely to buy natural yoghurt as they are to buy apricot yoghurt.

(v) Calculate the probability that the Dolans will buy natural yoghurt:

(1) the next week;

(2) two weeks after returning;

(3) three weeks after returning. [*5*]

(vi) Find an approximate value for the probability that the Dolans will buy strawberry yoghurt 30 weeks after returning. [*1*]

Question 5 – 12 marks

This question is intended to assess your understanding of birth and death processes.

You should be able to answer this question after working through Part I of Book 4.

(a) It is proposed to use an immigration–death process to model the arrivals and departures of visitors at a museum during opening hours.

(i) What assumptions about arrivals are made by using this model? [2]

(ii) Give two reasons why the model may not be suitable in practice. [2]

(b) The integer-valued random variable $X(t)$ denotes the number of individuals alive at time t in a simple birth process $\{X(t); t \geq 0\}$. A partial differential equation for $\Pi(s,t)$, the probability generating function of $X(t)$, is

$$\frac{\partial \Pi}{\partial t} = -\beta s(1-s)\,\frac{\partial \Pi}{\partial s}.$$

In Activity 4.1 of *Book 4*, you showed that this partial differential equation has the general solution

$$\Pi(s,t) = \psi\left(\frac{s}{1-s}\,e^{-\beta t}\right).$$

Suppose that $X(0)$, the number of individuals alive at time 0, is a random variable: $X(0)$ has the negative binomial distribution with range $\{3, 4, \ldots\}$ and parameters $r = 3$ and $p = 0.3$. Find the particular solution corresponding to this initial condition. Hence identify the probability distribution of $X(t)$ in this case, and find its mean. [8]

Question 6 – 18 marks

This question is intended to assess your understanding of birth and death processes.

You should be able to answer this question after working through Part I of Book 4.

(a) A population has size 0 at time 0. Arrivals occur according to a Poisson process with rate λ. In addition, from time to time, a 'catastrophe' occurs and the whole community is killed; these catastrophes occur according to a Poisson process with rate γ. (If there are no members, because there has been no arrival since the last catastrophe, then the population is not affected; its size remains 0.) Suggest an application different from that described in part (b) for which this process might provide a useful model. [*2*]

(b) During office hours, spam emails arrive in Darren's inbox according to a Poisson process with rate λ. Darren clears all the spam emails from his inbox from time to time according to a Poisson process with rate γ. At time 0, there are no spam emails in his inbox.

The number of spam emails in the inbox at time t is denoted $X(t)$, and the probability generating function of $X(t)$ is denoted $\Pi(s,t)$. You may assume that a partial differential equation for $\Pi(s,t)$ is

$$\frac{\partial \Pi}{\partial t} = \gamma - (\lambda + \gamma - \lambda s)\Pi.$$

(i) Write down the auxiliary equations for this partial differential equation. [*1*]

(ii) One solution of the auxiliary equations is $c_1 = s$. Show that a second solution can be written as

$$c_2 = (\gamma - (\lambda + \gamma - \lambda s)\Pi)e^{(\lambda+\gamma-\lambda s)t}.$$

(Hint: This is similar to Example 3.6 of *Book 4*.) [*4*]

(iii) Write down an equation satisfied by the general solution for Π, and hence find the particular solution corresponding to the given initial condition. [*6*]

(iv) Find $\lim_{t\to\infty} \Pi(s,t)$, and hence identify the equilibrium distribution of the number of spam emails in Darren's inbox. [*3*]

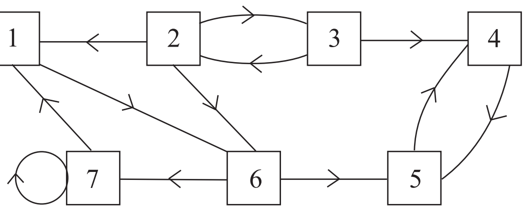
(v) If the rate at which Darren stops to clear the spam emails from his inbox doubles, how will the mean number of emails in his inbox change in the long run? [*2*]

Note that sometimes there will be more than one correct option for a particular question. Where this is so (and you will be told when this is the case), you should identify all the correct options.

Questions 1 to 3 are on Part III of Book 3.

Questions 1 to 3

The possible transitions between the states of a seven-state Markov chain are shown in the diagram below.



The communicating classes are $\{1, 6, 7\}$, $\{2, 3\}$ and $\{4, 5\}$.

1 Choose the option that gives a correct description of the class $\{1, 6, 7\}$.

2 Choose the option that gives a correct description of the class $\{2, 3\}$.

3 Choose the option that gives a correct description of the class $\{4, 5\}$.

Options for Questions 1 to 3

A Closed, recurrent, aperiodic

B Closed, transient, aperiodic

C Closed, recurrent, period 2

D Closed, transient, period 2

E Not closed, recurrent, aperiodic

F Not closed, transient, aperiodic

G Not closed, recurrent, period 2

H Not closed, transient, period 2

Questions 4 to 6

An immigration–birth process with arrival rate λ and birth rate β may be described by the probability statement

$$P(X(t+\delta t) = x+1 \,|\, X(t) = x) = (\lambda + \beta x)\delta t + o(\delta t).$$

Suppose that at time 0 the size of the population in an immigration–birth process is 1, and that $\lambda = 3\beta$.

4 Choose TWO options, one that gives the expected waiting time until the first event (arrival or birth), and one that gives the variance of this waiting time.

Options for Question 4

A $\dfrac{1}{3\beta}$ **B** $\dfrac{1}{4\beta}$ **C** $\dfrac{1}{5\beta}$

D $\dfrac{1}{9\beta^2}$ **E** $\dfrac{1}{16\beta^2}$ **F** $\dfrac{1}{25\beta^2}$

5 Choose the option that gives the expected waiting time until the population size reaches 3.

6 Choose the option that gives the variance of the waiting time until the population size reaches 3.

Options for Questions 5 and 6

A $\dfrac{9}{20\beta}$ **B** $\dfrac{5}{6\beta}$ **C** $\dfrac{7}{12\beta}$ **D** $\dfrac{41}{400\beta^2}$

E $\dfrac{81}{400\beta^2}$ **F** $\dfrac{13}{36\beta^2}$ **G** $\dfrac{5}{4\beta^2}$ **H** $\dfrac{9}{4\beta^2}$

Questions 7 to 9

Suppose that a simple birth–death process with $\beta = \frac{3}{7}\nu$ is initiated by two individuals. The embedded process $\{X_n; n = 0, 1, \dots\}$ describes the size of the population immediately after the nth change of state ($X_0 = 2$).

7 Choose the option that gives the value of the probability $P(X_{n+1} = x+1 \,|\, X_n = x)$ for $x = 1, 2, \dots$.

Options for Question 7

A $\frac{3}{10}$ **B** $\frac{3}{7}$ **C** $\frac{7}{10}$ **D** $\frac{3}{4}$

8 Choose the option that gives the probability (correct to four decimal places) that a simple birth–death process starting with two individuals and with these parameters will ever attain a population size of 5.

Options for Question 8

A 0.0652 **B** 0.1717 **C** 0.8283 **D** 0.9348

9 Choose the option that gives the expected number of events (births and deaths) until the population in a simple birth–death process starting with two individuals and with these parameters will eventually die out.

Options for Question 9

A 4.18 **B** 5 **C** 6 **D** 14

Questions 10 to 20 are on Part II of Book 4.

Questions 10 to 14

During the morning rush hour, arrivals at and departures from a newsstand by members of the public may be modelled as a simple queue. On average, three customers arrive every minute, and the mean time that the news vendor spends with each customer is 12 seconds.

10 Choose the option that gives the traffic intensity of this queue.

11 Choose the option that gives the proportion of people who arrive to find that there is no one else at the newsstand, and so receive immediate attention.

Options for Questions 10 and 11

A $\frac{1}{4}$ **B** $\frac{2}{5}$ **C** $\frac{3}{5}$ **D** $\frac{3}{4}$

12 Choose the option that gives the mean total time (in seconds) that a customer spends at the newsstand (waiting for service and receiving attention).

Options for Question 12

A 0.5 **B** 8 **C** 12 **D** 20 **E** 30

13 Choose the option that gives the proportion of customers (correct to four decimal places) who are at the newsstand for more than 15 seconds.

Options for Question 13

A 0.1353 **B** 0.2865 **C** 0.3935 **D** 0.6065

14 Choose TWO options, one that gives the mean length (in seconds) of an idle period for the news vendor and one that gives the mean length (in seconds) of a busy period.

Options for Question 14

A 8 **B** 12 **C** 16 **D** 20 **E** 30

Questions 15 to 17

Members of the public arrive at random at the rate of 3 per hour at an advice centre where there is one adviser on duty. If the adviser is busy, then arrivals form a queue. The time spent with the adviser is constant and equal to ten minutes.

15 Choose the option that gives the specification of the queue.

Options for Question 15

A $D/M/1$ **B** $M/D/1$ **C** $M/M/1$

D $M/U/1$ **E** $U/M/1$

16 Choose the option that gives the traffic intensity of the queue.

Options for Question 16

A $\frac{3}{10}$ **B** $\frac{1}{2}$ **C** 2 **D** $3\frac{1}{3}$

17 Choose the option that gives the mean equilibrium queue size (correct to three decimal places).

Options for Question 17

A 0.364 **B** 0.429 **C** 0.750 **D** 1.000

Questions 18 to 20

Patients arrive at random at the rate of 5 per hour at an emergency surgery where there is one doctor on duty. If the doctor is busy, then the patients form a queue. The time spent with the doctor has a uniform distribution, being anything between 5 minutes and 15 minutes.

18 Choose the option that gives the specification of the queue.

Options for Question 18

A $D/M/1$ **B** $M/D/1$ **C** $M/M/1$

D $M/U/1$ **E** $U/M/1$

19 Choose the option that gives the traffic intensity of the queue.

Options for Question 19

A $\frac{1}{2}$ **B** $\frac{5}{6}$ **C** 1.2 **D** 2

20 Choose the option that gives the mean number of patients in the surgery (correct to three decimal places), either consulting the doctor or waiting to see her.

Options for Question 20

A 0.924 **B** 2.917 **C** 3.090 **D** 5.000 **E** 13.333

Note that sometimes there will be more than one correct option for a particular question. Where this is so (and you will be told when this is the case), you should identify all the correct options.

Questions 1 to 6 are on Part III of Book 4.

Questions 1 to 4

Four students in a group of seven friends sharing a house are suffering from an infectious disease that is incurable but not serious; the other three are susceptible. The spread of the disease through the household is to be modelled as a simple stochastic epidemic, with contact rate $\beta = 0.24$ per day.

1 Choose the THREE options that give the values of β_4, β_5 and β_6.

Options for Question 1

A 0 **B** 0.2 **C** 0.24

D 0.36 **E** 0.4 **F** 0.48

2 Choose the option that gives the expected waiting time in days (correct to two decimal places) until a fifth person catches the disease.

Options for Question 2

A 2.08 **B** 2.50 **C** 4.17 **D** 5.00

3 Choose the option that gives the expected waiting time in days (correct to two decimal places) until all seven friends have contracted the disease.

Options for Question 3

A 1.12 **B** 6.67 **C** 8.75 **D** 9.44 **E** 17.50

4 Choose the option that gives the standard deviation of the waiting time in days (correct to two decimal places) until all seven friends have contracted the disease.

Options for Question 4

A 4.86 **B** 5.29 **C** 8.75 **D** 23.61 **E** 27.95

Questions 5 and 6

A deterministic general epidemic model is used to describe the progress of an infectious disease through a closed community of 26 people. Initially, there are two infectious individuals, and everyone else is susceptible. The contact rate β is 3 per week, and the removal rate γ is 1.8 per week, so the value of the epidemic parameter ρ is 15.

5 Choose the option that gives the maximum number of people who are simultaneously infectious (correct to two decimal places).

Options for Question 5

A 2.75 **B** 3.95 **C** 7.42 **D** 7.94

6 Choose the option that gives the number of people who survive the epidemic without catching the infectious disease (correct to two decimal places).

Options for Question 6

A 0.36 **B** 6.57 **C** 7.65 **D** 8.59

Questions 7 to 12 are on Part IV of Book 4.

Questions 7 to 12

The life table function for the members of a stationary population is

$$Q(x) = \left(1 - \frac{x}{100}\right)^4, \quad 0 \leq x < 100,$$

where x is measured in years.

7 Choose the option that gives the mean age in years at which death occurs (correct to one decimal place).

8 Choose the option that gives the median lifetime in years of members of the population (correct to one decimal place).

Options for Questions 7 and 8

A 12.9 **B** 15.9 **C** 16.7 **D** 20.0 **E** 33.3

9 Choose the option that gives the proportion of individuals who live for more than 90 years.

10 Choose the option that gives the proportion of individuals who live for less than 10 years.

11 Choose the option that gives the proportion of members of the population who at any time are aged 90 years or more.

12 Choose the option that gives the proportion of members of the population who at any time are less than 10 years old.

Options for Questions 9 to 12

A 0.000 01 **B** 0.0001 **C** 0.3439 **D** 0.409 51

E 0.590 49 **F** 0.6561 **G** 0.9999 **H** 0.999 99

Question 13

A fat mouse **bbDd** (see Example 1.5 in *Book 5*) is crossed with a 'normal' mouse **BbDd**. Choose the option that gives the proportion of the offspring of this mating that you should expect to be fat.

Options

A $\frac{1}{4}$ **B** $\frac{3}{8}$ **C** $\frac{1}{2}$ **D** $\frac{5}{8}$ **E** $\frac{3}{4}$

Questions 14 to 17

In a large population of animals reproducing in discrete generations by a process of random mating, the fur characteristics are controlled by a single gene. Doubly dominant animals **AA** and heterozygotes **Aa** have glossy fur; homozygous recessives **aa** have dull fur. In the population as a whole, 64% of animals have dull fur.

14 Choose the option that gives the proportion of animals in the population that are 'pure' glossy-furred (**AA**).

Options for Question 14

A 0.04 **B** 0.16 **C** 0.32 **D** 0.36

15 Choose the option that gives the proportion of animals with glossy fur that are 'pure' glossy-furred (**AA**).

16 Choose the option that gives the proportion of animals with glossy fur that are genotype **Aa**.

Options for Questions 15 and 16

A $\frac{1}{25}$ **B** $\frac{1}{9}$ **C** $\frac{8}{25}$ **D** $\frac{1}{2}$ **E** $\frac{8}{9}$

17 Two glossy-furred animals are selected at random from the population and mated. Choose the option that gives the proportion of their offspring that you should expect to have dull fur (correct to four decimal places).

Options for Question 17

A 0.1975 **B** 0.2500 **C** 0.4096

D 0.6400 **E** 0.7901

Questions 18 to 20

The wing type (plain or spotted) of a particular species of insect is determined by a sex-linked gene with two alleles, **C** (which is dominant) and **c**. Spotted wings are recessive to plain wings.

A plain-winged male is mated with a heterozygous plain-winged female.

18 Choose TWO options, one that gives the proportion of female offspring of this mating that have plain wings, and one that gives the proportion of male offspring of this mating that have plain wings.

Options for Question 18

A 0 **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$ **E** 1

One male and one female are chosen at random from all the offspring of this mating and are themselves mated.

19 Choose the option that gives the proportion of male offspring of this second mating that will have spotted wings.

20 Choose the option that gives the proportion of female offspring of this second mating that will have plain wings.

Options for Questions 19 and 20

A $\frac{1}{8}$ **B** $\frac{1}{4}$ **C** $\frac{3}{8}$ **D** $\frac{1}{2}$

E $\frac{5}{8}$ **F** $\frac{3}{4}$ **G** $\frac{7}{8}$ **H** 1

Questions 1 to 8 below, on Parts II, III and IV of *Book 4*, and *Book 5*, form tutor-marked assignment M343 03. The marks available for the questions are indicated at the beginning of each question.

Please send all your answers to your tutor, along with an appropriately completed assignment form (PT3). Be sure to fill in the assignment number on this form as

$\boxed{\text{M343 03}}$.

Question 1 – 10 marks

This question is intended to assess your understanding of the $M/M/n$ queue.

You should be able to answer this question after working through Part II of Book 4.

Six cashiers are on duty in a department store where customers may be assumed to arrive independently and at random, at the rate of 48 per hour. If a cashier is free, then an arriving customer receives immediate attention; otherwise, a central queue is formed. The service time for each cashier may be assumed to be exponentially distributed with mean 2.5 minutes.

(a) Calculate the traffic intensity ρ. *[1]*

Assume that the queue is in equilibrium.

(b) Show that the probability that at any particular time all six cashiers are idle is $\frac{5}{37}$. *[4]*

(c) For what proportion of the time are exactly four of the six cashiers busy? *[1]*

(d) What proportion of customers receive immediate attention? *[2]*

(e) What proportion of customers arrive to find all six cashiers busy and one person waiting to be served? *[2]*

Question 2 – 10 marks

This question is intended to assess your understanding of the stochastic general epidemic model.

You should be able to answer this question after working through Part III of Book 4.

A stochastic general epidemic model is to be used to describe the progress of an infectious disease through a closed community of 26 people. Initially, there are two infectious individuals; the others are all susceptible. The contact rate β is 3 per week, and the removal rate γ is 1.8 per week.

(a) Show that the epidemic parameter ρ is equal to 15. *[1]*

(b) Draw diagrams showing all the epidemic paths for which at most one person catches the disease before the epidemic dies out. Calculate the probability that of the initial 24 uninfected susceptibles, at most one person catches the disease before the epidemic dies out. *[8]*

(c) Calculate an approximate value for the probability that there will be a minor outbreak of the disease. *[1]*

Question 3 – 15 marks

This question is intended to assess your understanding of models for stationary populations and stable populations.

You should be able to answer this question after working through Part IV of Book 4.

All that is known about a certain stationary population of animals is that the age-specific death rate is given by the formula

$$h(x) = \frac{4x^3}{625 - x^4}, \quad 0 \le x < 5,$$

where x is measured in years.

(a) Draw a sketch of $h(x)$, and hence describe the way in which death rates change with age. *[2]*

(b) Find the life table function $Q(x)$ for animals in this population. *[3]*

(c) Find the expectation of life at birth, e_0. *[2]*

(d) At any time, find the mean age of animals in the population. *[3]*

(e) Sketch $g(x)$, the p.d.f. of the age distribution of animals in the population. *[1]*

(f) *No calculations are required for this part of the question.*

Now suppose that a population with the life table function $Q(x)$ that you obtained in part (b) is stable but not stationary. Describe briefly how the age distribution of the population would differ from that of a stationary population with the same life table function if the population is (i) growing, (ii) declining. Draw diagrams to show what the p.d.f. of the age distribution might look like in each case. *[4]*

Question 4 – 15 marks

This question is intended to assess your understanding of population genetics.

You should be able to answer this question after working through Part I of Book 5.

Suppose that the numbers of people in the three blood groups **M**, **MN**, **N** in generation 0 of a population are in the proportions $0.45 : 0.5 : 0.05$, and that the population reproduces generation by generation by a process of random mating. (The **M-N** blood group system is described in Examples 1.3 and 2.4 of *Book 5*.)

(a) What proportion of members of generation 1 and of subsequent generations will be in each of the blood groups **M**, **MN** and **N**? *[3]*

Assume that the population has been reproducing for several generations. Colin, whose blood group is **MN**, and Delia, whose blood group is not known, have a son Harry.

(b) Calculate the probabilities of the different blood groups for Harry. *[6]*

(c) A test shows that Harry's blood group is **M**. Calculate the probabilities of the possible blood groups for his mother Delia. *[6]*

Question 5 – 10 marks

This question is intended to assess your understanding of discrete-time renewal processes.

You should be able to answer this question after working through Section 6 of Book 5.

The waiting time T between successive occurrences of an event E in a discrete-time renewal process has the probability distribution

$$P(T = 1) = 0.6, \quad P(T = 3) = 0.4.$$

(a) Find the generating function $U(s)$ for this process, and hence find the probabilities u_1, u_2, u_3 and u_4. [*5*]

(b) Write down the p.g.f. of W_8, the waiting time to the eighth event. Hence or otherwise calculate the value of $P(W_8 = 14)$. [*4*]

(c) Find an approximate value for the probability that an event occurs at time 400. [*1*]

Question 6 – 15 marks

This question is intended to assess your understanding of renewal processes in continuous time.

You should be able to answer this question after working through Sections 7 and 8 of Book 5.

The lifetime T (in weeks) of the battery in a certain model of portable radio may be described in terms of its hazard function

$$h(t) = \tfrac{1}{9}t, \quad t \geq 0.$$

(a) Find the survivor function $Q(t)$. [*2*]

(b) Write down the c.d.f. of T, and use Table 9 in the *Handbook* to identify the distribution of T. Hence show that the mean lifetime of these batteries is approximately 3.760 weeks, and calculate the variance of the lifetime. [*3*]

(c) What proportion of batteries last for less than four weeks? [*1*]

(d) Find the median lifetime of these batteries. [*2*]

(e) Decide whether the batteries are NBU, NWU or neither, justifying your answer. [*4*]

Assume that whenever a battery fails, it is immediately replaced. Robert has a radio containing one of these batteries. Suppose that, at some time when the radio is quite old, Robert gives it to his brother Tim.

(f) Find the mean total lifetime of the battery in the radio when Robert gives the radio to Tim. Hence find the mean residual lifetime of this battery. [*2*]

(g) If Tim keeps the radio for a year (52 weeks), how many times should he expect to replace the battery in that time? [*1*]

Question 7 – 15 marks

This question is intended to assess your understanding of ordinary Brownian motion.

You should be able to answer this question after working through Part III of Book 5.

An animal leaves its burrow and moves up and down a north–south hedgerow at the edge of a field, foraging for food. Its distance in metres north of its burrow after t minutes is denoted $X(t)$, and may be reasonably modelled as ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = 3.2\,\mathrm{m}^2$ per minute.

(a) Calculate the probability that after five minutes the animal will be more than 5 metres north of its burrow. *[3]*

(b) Calculate the probability that after five minutes the animal will be within 5 metres of its burrow. *[1]*

(c) There is a beech tree 20 metres north of the burrow. Calculate the probability that it will be more than an hour before the animal reaches the tree. *[3]*

(d) One morning, the animal leaves its burrow at 8 am. Suppose that the animal is observed at the beech tree at 10 am. Calculate the probability that at 10.30 am it will be south of its burrow. *[4]*

(e) On another morning, the animal is observed to be level with its burrow at 11 am. Use the numbers 1.0418, −0.8050, 0.5349, which are random observations from the standard normal distribution $N(0, 1)$, to simulate its position at 20-minute intervals for the next hour. Give the distances north of the burrow to the nearest centimetre. *[4]*

Question 8 – 10 marks

This question is intended to assess your understanding of geometric Brownian motion.

You should be able to answer this question after working through Part III of Book 5.

The height above sea level of a migrating bird (of a certain species) when crossing the English Channel fluctuates in such a way that the random variable

$$H(t) = \frac{\text{height at time } t}{\text{height at time } 0}$$

may be modelled as geometric Brownian motion, derived from ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = 0.2$ per hour through the relationship $H(t) = \exp X(t)$. A bird crosses the English coast at time 0 at a height of 40 metres above sea level.

(a) Calculate the probability that after half an hour the bird will be less than 30 metres above sea level. *[4]*

(b) When the bird crosses the French coast two hours after leaving the English coast, it is observed to be 20 metres above sea level. Calculate the probability that when it passed a ship half an hour after leaving the English coast, it was less than 30 metres above sea level. *[6]*