

Question 2 (Unit GR2)

This question tests your ability to

- investigate and identify a particular group of small order;
- determine whether or not two direct products of cyclic groups are isomorphic;
- determine whether or not a direct product of cyclic groups is cyclic;
- find generating elements for a direct product of cyclic groups.

(a) Consider the matrices

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

in the multiplicative group of non-singular 2×2 matrices with entries in \mathbb{R} . Let G be the group generated by A and B , which you may assume has order 8.

- List the non-identity elements (other than A and B) of G , both as explicit 2×2 matrices and as products involving A and B .
- Write down the order of each of the non-identity elements of G .
- Since G has order 8, it is isomorphic to one of the groups

$$C_8, C_2 \times C_4, C_2 \times C_2 \times C_2, D_8, Q_8.$$

Decide which one, giving reasons for your answer.

- Show that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{15} \times \mathbb{Z}_{275}$ and $\mathbb{Z}_{75} \times \mathbb{Z}_{495}$ are isomorphic.
- Decide whether each of the following groups is cyclic or non-cyclic, giving reasons for your answer.

$$A = \mathbb{Z}_{44} \times \mathbb{Z}_9, \quad B = \mathbb{Z}_{26} \times \mathbb{Z}_{36} \quad \text{and} \quad C = \mathbb{Z}_{11} \times \mathbb{Z}_{12} \times \mathbb{Z}_{13}.$$

- Find two elements a and b of the group $H = \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_6$ such that $H = \langle a, b \rangle$.

$$\cos 2\theta = \frac{\sqrt{3}}{2}$$

$$\sin 2\theta = \frac{1}{2}$$

$$\sin \theta = \frac{1}{2}$$

$$\cos \theta = \frac{\sqrt{3}}{2}$$

$$\cos 2\theta = -\frac{1}{2}$$

$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

$$120$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$0 = y + \frac{dy}{dx}$$

$$3 = y + \frac{dy}{dx}$$