PHYSICS 306 WINTER SESSION - 2013

HOMEWORK 4

1. Consider a physical system described by a Lagrangian L. The Hamiltonian of the system is found by expressing the quantity $\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ in terms of the canonical variables (q, p). From the Hamiltonian, using Lagrange's equations, derive Hamilton's equations of motion.

2. A dynamical system has the Lagrangian

$$L = \dot{q}_1^2 + \frac{\dot{q}_2^2}{a + bq_1^2} + k_1 q_1^2 + k_2 \dot{q}_1 \dot{q}_2$$

where a, b, k_1 and k_2 are constants.

From the Lagrangian, find the conjugate momenta p_1 and p_2. Find the Hamiltonian. Write Hamilton's equations for this system.

3. Consider a coplanar double pendulum (Problem 1 of Section 5 in textbook). From the Lagrangian, find the conjugate momenta p_{ϕ_1} and p_{ϕ_2}. Find the Hamiltonian. Write Hamilton's equations for this system.

4. Two masses m_1 and m_2 are connected by a weightless string of fixed total length l. Mass m_1 moves on a frictionless table, which has a small hole cut into it. Mass m_2 hangs down vertically from this hole. Assume that m_2 can only move in the vertical direction, so the problem has two degrees of freedom.

a. Assuming that the acceleration of gravity is g, find the Lagrangian using plane polar coordinates (r, ϕ).

b. Find the Hamiltonian and the Hamilton equations for this system.

c. Find the total angular momentum. Prove that the total angular momentum is equal to the conjugate momentum p_ϕ. From Hamilton's equations, show that p_ϕ is conserved.

d. Is there a case where the motion of mass m_1 is circular? For this circular orbits, find the radius r as a function of the angular frequency ω.

5. A particle of mass m slides under the action of gravity and without friction on a wire shaped into a parabola. We choose x to be the generalized coordinate. The parabola has

the shape $y = \frac{x^2}{2}$. Show that the Lagrangian can be written as:

$L = \frac{1}{2}\left(1 + x^2\right)\dot{x}^2 - \frac{1}{2}mgx^2.$

Find the Hamiltonian. Write Hamilton's equations.

6. Consider a relativistic particle in a Coulomb field:

$L = -mc^2 \left(1 - \frac{v_x^2 + v_y^2 + v_z^2}{c^2}\right)^{\frac{1}{2}} - \frac{kee'}{r}.$

From the Lagrangian, find p_x, p_y and p_z. Show that the Hamiltonian takes the form:

$H = mc^2 \left(1 + \frac{p_x^2 + p_y^2 + p_z^2}{m^2c^2}\right)^{\frac{1}{2}} + \frac{kee'}{r}.$

7. Problems 1 and 3 of Section 40.

8. Prove that the Poisson brackets obey the Jacobi identities.

9. Prove that if $f(q, p, t)$ and $g(q, p, t)$ are two integrals of motion, then their Poisson bracket is also an integral of motion (Poisson's theorem).

10. Prove that Poisson brackets are invariant under canonical transformations.

11. Prove that for time-independent canonical transformations the new Hamiltonian equals the old Hamiltonian expressed in terms of the new canonical variables:

$H(Q, P) = H(q(Q, P), p(Q, P)).$

12. Consider the one-dimensional harmonic oscillator described by the Hamiltonian:

$H = \dfrac{p^2}{2m} + \dfrac{kq^2}{2}.$

Find a canonical transformation such that the Hamiltonian becomes a linear function of the new canonical momentum. Prove that the transformation found is canonical.

Write Hamilton's equations. Solve Hamilton's equations for the new canonical variables.

13. Consider an isolated conservative system of two particles with a potential energy depending only on the distance between the particles.

Write the Hamiltonian for this system. Find a canonical transformation that reduces the two-body problem to an effective one-body problem (a transformation for which the equations for the relative motion and the motion of the centre of mass are separated). Prove that the transformation you found is indeed canonical.

Write the Hamiltonian in terms of the new canonical variables. Write Hamilton's equations for the new canonical variables.

14. Problems 1, 2 and 3 of Section 42.

15. Find under what conditions: $Q = \dfrac{\alpha p}{x}$, $P = \beta x^2$, where α and β are constants, represents a canonical transformation.

16. Using the fundamental Poisson brackets, find the values of α and β for which the equations: $Q = q^\alpha \cos(\beta p)$, $P = q^\alpha \sin(\beta p)$, represents a canonical transformation.

17. Determine whether the transformation:

$$x = \frac{1}{\alpha}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)$$
$$y = \frac{1}{\alpha}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)$$
$$p_x = \frac{\alpha}{2}\left(\sqrt{2P_1}\cos Q_1 - Q_2\right)$$
$$p_y = -\frac{\alpha}{2}\left(\sqrt{2P_1}\sin Q_1 - P_2\right)$$

(where α is some fixed parameter) is canonical.