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Concept of Limit in Real Analyses

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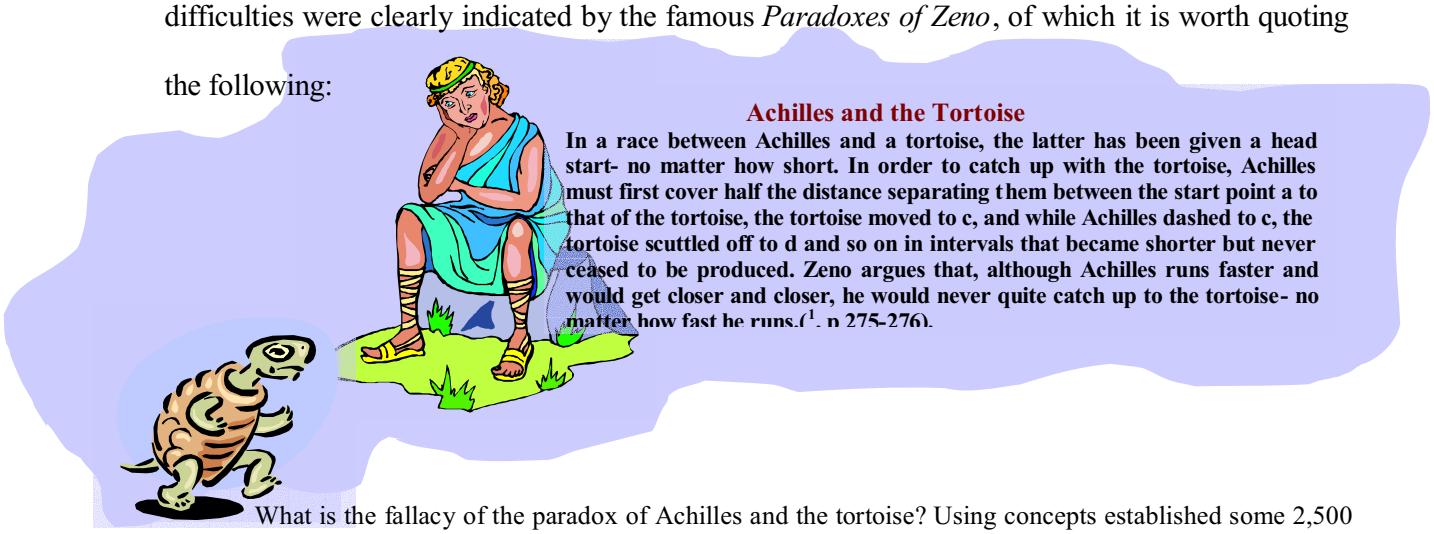
Introduction

There is probably no other instance in human intellectual history in which so much time and effort was spent merely to reach a satisfactory definition as that for limit. The concept is very closely related with two other fundamental concepts of mathematics and exists alongside both infinity and continuity. The Greek scholars were the first who seriously considered problems of continuity and infinity based on the concept of 'number'. Various attempts were made by them to include the concept of number into geometry. Being able to construct a line segment of any rational *m/n* length $(m,n \in Z, n \neq 0)$, the Greeks discovered around 400 B.C. that a diagonal of a unit triangle is an irrational number, which falls out of the number concept. Questions of continuity and infinity seem to have represented a complete mystery to the Greek scholars. The difficulties were clearly indicated by the famous *Paradoxes of Zeno*, of which it is worth quoting the following:



Achilles and the Tortoise

In a race between Achilles and a tortoise, the latter has been given a head start- no matter how short. In order to catch up with the tortoise, Achilles must first cover half the distance separating them between the start point a to that of the tortoise, the tortoise moved to c, and while Achilles dashed to c, the tortoise scuttled off to d and so on in intervals that became shorter but never ceased to be produced. Zeno argues that, although Achilles runs faster and would get closer and closer, he would never quite catch up to the tortoise- no matter how fast he runs.([1], p 275-276).

What is the fallacy of the paradox of Achilles and the tortoise? Using concepts established some 2,500 years after Zeno, here is the explanation why Achilles can finally catch up to and pass the tortoise:

Although the number of time intervals is infinite, the total amount of time is not necessarily infinite. Suppose the tortoise is given a head start of 3 meters and advances at the speed of 3m/s and Achilles ambles along at 6 m/s, Achilles will catch up to the tortoise at the end of (see Appendix 1):

(1) $$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{1/2}{1-1/2} = 1\,\text{sec} \quad .$$

The space above introduced by '**…**' stands for the infinite number of decreasing fractions called a **sequence**, that add up to 1 second. Although it seems not so difficult to express every term of the sum with respect to its place - no matter how far it is- the entire process of computation of the sum of the infinite number

of terms is not clear. On the other hand, even if one could intuitively estimate, that the terms of the above sequence tend to zero (converge to zero), as one chooses a term far enough, it is still unclear, how to justify one's intuitive guess, because he/she might have intuition different from others. The space '**...**' in (1) reflects the distance between first attempts of ancient scholars to halt 'leak' of information about incompleteness of rational numbers they discovered, and the modern era development of old concepts such as differential and integral calculus, which solved and interpreted ancient and modern enigmas.

Amongst the earliest and most significant contributors to rigor in calculus was A. Cauchy. He explained the meaning of the above expression 'tends to zero as the term is far enough' in following terms:

' When the successive values attributed to a variable approach infinitely a fixed value so as to end by differing from it by as little as one wishes, this fact is called the limit of the others.' ([2], p. ***x***)

This definition seems excessively vague from our viewpoint: the phrases '*successive values*', '*approach indefinitely*', '*as little as one wishes*' are suggestive rather then mathematically precise. Therefore A. Cauchy's definition needed to be refined in terms of formal mathematical language and this was done by H.E. Heine forty three years after the first publication of the above definition of limit. Heine defined the limit of a function *f(x)* at x_0:

'*If given any* ε *, there is an* η_0 *such that for* $0 < \eta < \eta_0$ *the difference* $f(x \pm \eta) - L$ *is less in absolute value than* ε *, then L is the limit of f(x) for* $x = x_0$.'

This statement, which is now the accepted definition of limit, is absolutely unambiguous. With minor modifications, it applies to many other kinds of limiting processes, including sequences and series of numbers and functions, functions of several variables, complex functions etc. The paradox of Zeno regarding motion disappears once the definition of continuity based on Heine's definition of limit is understood. This also led to clear definitions of number, continuity, and derivative enabling nineteenth century scientists to provide a logically precise development of calculus. With an instrument as powerful as calculus, modern mathematicians solved problem of estimating volumes of solids formulated by Archimedes.. We strongly believe that it is impossible to teach someone this concept. However those lucky to touch it may feel as great a pleasure as those who understand the Bach harmonies. In this essay we discuss only three applications of the concept, namely, the limit of sequence, the limit of series and the limit of function of one real variable supporting our reasoning with some samples. As the amount of words of the essay is strictly LIMITED we are not discussing a concept of infinity or continuity, which are based on the concept of limit. However when necessary we give outline of the former without further contemplations or speculation.

Limit of Sequence

$\{x_n\}_{n=1}^{\infty}$ is a function defined on a set of real numbers for all positive integers.

The intuitive definition of the sequence is already outlined in the Introduction. Roughly speaking,

Definition 1: *$\{x_n\}_{n=1}^{\infty}$ has a limit $L \subset R$ if (x_n-L) becomes arbitrary small for all sufficiently large values of n.* In this case we write

(2) $$\lim_{n \to \infty} x_n = L \cdot$$

From this crude description, we would expect that the sequence 1,1,… has the limit 1, whereas the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ has a limit 0 while the sequence 1,-2,3,-4,… does not have a limit. On the other hand, our intuition is not sharp enough to deduct, whether the sequence $\{n \sin(\pi n)\}_{n=1}^{\infty}$ has a limit and to compute the limit if there is one. Even a relatively simple sequence as , -1,1, -1,…, so-called *oscillated* sequence, could lead to the wrong intuitive conjecture, that it has two limits. So, we need an accurate definition of the 'limit of a sequence' based on which we would become capable to predict for any given sequences the existence of its limit and to evaluate it.

We emphasise that limit *L* should be a real number. Formally, **Definition 1** means, that

Definition 2: (3) $\forall \varepsilon > 0 \;\; \exists N : \forall n > N \therefore |x_n - L| < \varepsilon$.

We can interpret it as follows: the proof that *L* is the limit of a given sequence $\{x_n\}_{n=1}^{\infty}$, consists upon being given an $\varepsilon > 0$ of finding a value of *N*, such that the inequality $|x_n - L| < \varepsilon$ holds for all values of *n* except at most a finite number, namely $n = \overline{1, N-1}$. The value of *N* will, in general, depend on the value of ε. **Figure 1** illustrates **Definition 2**. All of the x_n, except at most a finite number of terms, must be inside the parentheses.

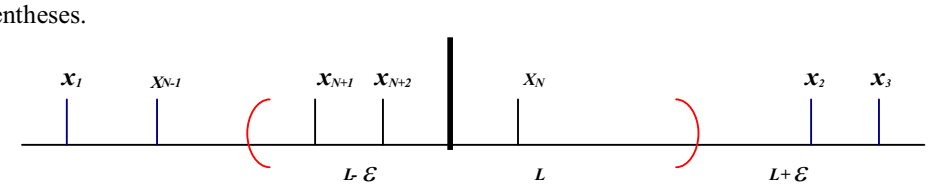


Figure 1

How **Definition 2** works in practice?

Example 1: Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$, which can be expressed as x_n=1/n (n=1,2,3,...). We already made a conjuncture that $\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{1}{n} = 0$. Let us prove it.

Following **Definition 2** for given $\varepsilon > 0$ we must find N so that for all n>N: $|x_n - 0| < \varepsilon$.

Substitution of $x_n=1/n$ into the last inequality leads to

$\left|\frac{1}{n} - 0\right| < \varepsilon$, or, considering $n>0$ (4) $\quad n = \frac{1}{\varepsilon}$.

Thus if we choose N so, that $1/N < \varepsilon$, then certainly (4) will hold since $\frac{1}{n} \le \frac{1}{N}$ for all n larger than N.

Now $\frac{1}{n} < \varepsilon$ iff $N > \frac{1}{\varepsilon}$. Hence if we take any integer N such that $N > \frac{1}{\varepsilon}$, then (3) will hold for the given sequence with $L=0$. This proves that $\lim_{n \to \infty} x_n = 0$ <u>although not one term of the sequence is equal to zero</u>.

Example 2 Examining the sequence $x_n=1$(n=1,2,3,…) in terms of **Definition 2** we can prove that our guess $\lim_{n\to\infty} x_n = 1$ was correct. Really, if $L=1$ then $|x_n - L| = |1-1| = 0$ for any $\varepsilon > 0$. So in this case (3) <u>always holds and N does not depend on ε</u>.

Example 3. Consider the sequence 1,2,3,…, which can be expressed as $\{x_n\}_{n=1}^{\infty} = n$ .It can be proved by contradiction using the same ε-N method, which leads to the statement that the sequence 1,2,3,… has no limit or, that $\{x_n\}_{n=1}^{\infty}$ *tends to infinity* when n tends to infinity or *diverges to infinity*.

<u>*Infinity* is certainly not a number. Moreover, it is not a quantative concept. It is a quality of increase beyond bound.</u> Although the concept of infinity is difficult to grasp we can define it as not finite, contrary to finite, which is completely determinable by counting or measurement. Following the Galileo statement that there are as many squares as there are natural numbers, G. Cantor proved that the set of all integers, the set of all natural numbers, the set of all rational numbers and the set of all algebraic numbers are equivalent to the set of all natural numbers as they can be put in one-to-one correspondence with the infinitude of natural numbers. Following this concept we may think of all divergent to infinity sequences as having the 'same *manyness* of elements' as, by the definition, each sequence has one- to one correspondence with the set of all positive integers, hence belong to 'aleph null' set of infinity ([1]p.258-264). Remembering though, that infinity cannot be

expressed by any number (other things, like motherhood, happiness, faith belong to the qualitative category of concepts, that humans were hopelessly trying to describe by quantity), we discourage the idea to resolve indeterminate problems of $\frac{\infty}{\infty}$; $\infty - \infty$; substituting each term of the former by equal numbers in case of; for example, { $\frac{a^n}{n!}$ } simply because there are no numbers equal to infinity.

Coming back to **Example 3** whatever large *number* we choose there are always terms, which would exceed it. This reasoning seems to breach **Definition 2**. Really, we determine infinite as not a number, hence in this case the inequality $|x_n - L| < \varepsilon$ makes no sense. So **Definition 2** needs amendments such that there would not be a need to use the 'suspicious' concept in notation. We have already proved, that a sequence may have a limit, which is different to any of its terms, so the fact that '∞'is not a number should not contradict the perception of a limit. Following experience with the sequence $\{x_n\}_{n=1}^{\infty}$=*n* we know, that

Definition 3 *For any given positive number M, there is an integer N (depending on M) such, that* $x_n > M$ *for every* $n \geq N$.

This definition binds concepts of convergence to a finite limit as to a real number and divergence to infinity.

. Is a sequence whose terms get ' too big' as in **Example 3**, the only kind, which does not have a limit? We already consider a 'suspicious' example $x_n = 1,-1,1,-1,...$ Let us suppose that it has a limit, so $\lim_{n\to\infty} x_n = L$. **Definition 2** has to be satisfied for any $\varepsilon > 0$. Let us choose $\varepsilon = \frac{1}{2}$. Following (3) there would be a positive integer *N* such as $|(-1)^n - L| < \frac{1}{2}$ *for* $n \geq N$.

If *n* is even then the last expression means:/1-*L*/< ½ and for *n* odd it is /-1-*L*/<*1*. This implies that *L* should be less than half unit from 1 and less that half unit from –1, which is impossible. So by contradiction we proved that the sequence 1,-1,1,-1,... has no limit though the terms of the sequence all have absolute value 1 and hence are not 'too big'. It is worth noticing, that the initial guess, that there may be two limits of the above sequence is proved to be wrong.

The last example illustrates one very important property of limit of sequence: *a sequence has at most one limit*.

Limit of Series

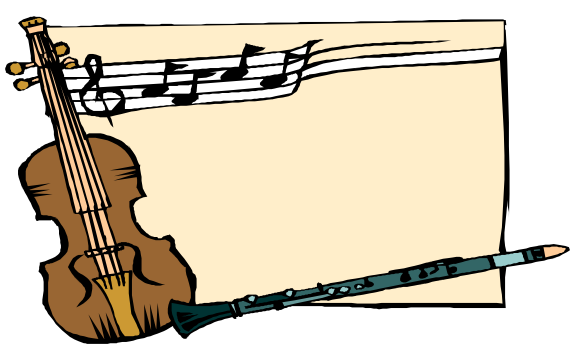
Another very important application of the concept under discussion is the limit of series of real numbers, which already appeared in the *Zeno Paradox* (see formula (1)). Formally

Definition 4 *Infinite series of real numbers can be defined as an ordered pair* $\left\langle \{a_n\}_{n=1}^{\infty}, \{s_n\}_{n=1}^{\infty} \right\rangle$, *where* $\{a_n\}_{n=1}^{\infty}$ *is a sequence and* $\{s_n\}_{n=1}^{\infty} = \sum_{i=1}^{n} a_i$ *is called the* n^{th} *partial sum of the series* that forms a new sequence $\{s_n\}$, which can either converge to the limit S, if the limit does exist $\lim_{n\to\infty} s_n = S$, or diverge, if $\lim_{n\to\infty} s_n$ does not exist or if s_n diverges to ∞.

From the above definition it is easy to deduce, that the behaviour of series depends on the behaviour of the sequence of its partial sums. Moreover, we can make a conjuncture, that for partial sums to converge to a real number, the limit of the term a_n ought to become 'smaller and smaller', or, formally, tend to zero as n tends to infinity. Unfortunately, this property is not sufficient to ensure, that $\{s_n\}_{n=1}^{\infty} = \sum_{i=1}^{n} a_i$ be convergent. On the other hand, it can be used to determine a divergence of a series if $\lim_{n\to\infty} a_n \neq 0$.

Following example illustrates the point of the discussion.

Consider series $s_n = \sum_{i=1}^{\infty} \frac{1}{n}$. From Chapter 1 we know that $\lim_{n\to\infty} \frac{1}{n} = 0$. However, it can be proved, that s_n diverges when $n \to \infty$. This is an example of a *harmonic series*, which terms are reciprocal to arithmetic sequences. In music, vibrating strings of the same material and with equal diameter, equal torsion and equal tension and whose lengths are proportional to terms in a harmonic sequence generate harmonic tones. Referring to A. Pushkin his personage Salieri may not be too wrong trying to test ' harmonies by algebra'. *Knowing much more about harmonic series than people in XVIII century, we would rather opt for the real analyses than for algebra.*



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Limit of Function of the Real Variable

Chapters 1 and 2 were concerned in mapping discrete integers in set of real numbers. We are now interested in mapping an interval *A* of real numbers in the set of real numbers *B*.

Definition 4 *We say that f is a function from A to B if for every* $a \in A$ *there is exists a unique* $b \in B$ *such that* $(a,b) \in f$ *, or* $f(a) = b$

By contrast to a rational number, which caused concern to the Greek mathematicians due to the lack of completeness, real numbers *R* possesses the completeness property. This implies that real numbers in an interval A cannot be ordered by corresponding integer as we do for sequences. We order rational numbers by comparison of their values. So the definition we used for the limit of sequences and series needs adoption with respect to the completeness property of the former. Let $a \in R$ and let *f* be a real-valued function whose domain includes all points in some open, interval (*a-h, a+h*) except, possibly, the point *a* itself.

Definition 5 *We say that f(x) approaches L (where* $L \in R$*) as x approaches a if given* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *as* $0 < |x - a| < \delta$ *.It is worth noting that point a need not be in the domain of f.* Thus well known $\lim_{n \to 0} \frac{\sin x}{x} = 1$ although the function is not defined in *n=0*.

Definition 5 can be graphically illustrated (see **Figure 2**).

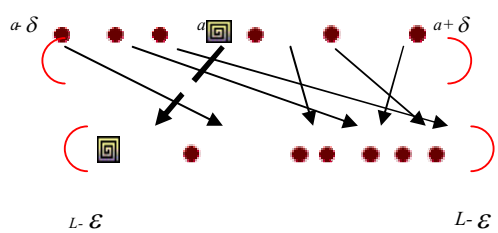


Figure 2

In order for *f(x)* to approach *L* as *x* approaches *a* the following must be true: given any ε parentheses about *L* there must exist δ parentheses about *a* such that every arrow which begins inside the δ parentheses (except, possibly the arrow if there is one, that starts at *a*) must end inside the ε parentheses.

Roughly speaking, the following can be seen on the graph of a function *f* such that (Figure 3): as the *x* coordinate of a moving point of the graph gets close to *a* (from either the right or the left), the height *f(x)* of the point heads toward *L*. Thus both lines introduce functions *y=f(x)* (in red) and *y=g(x)* (in green) satisfy $\lim_{x \to a} f(x) = L$. Moreover, following Definition 5 even if *f(x)* and

g(x) are defined such that *f(2)* and *g(2)* are not equal to *3*, both functions still have *3* as their limit in x=2.

Figure 3

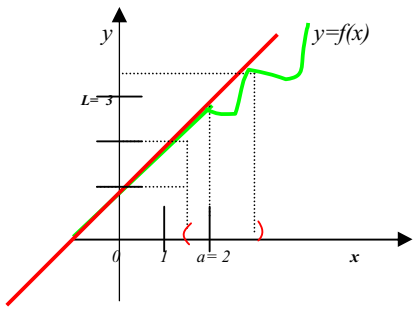
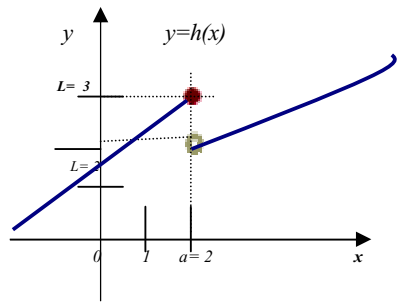


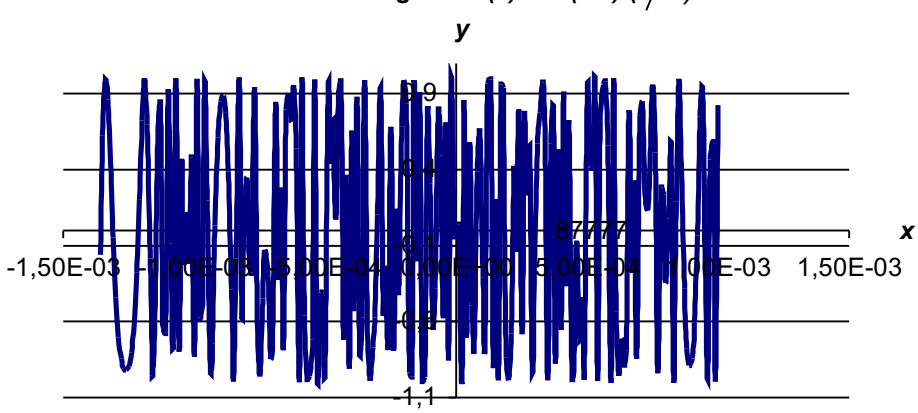
Figure 4



On the other hand, the function *y=h(x)* in **Figure 4** has no limit at a=2 because *h(x)* gets close to 3 when x gets close to a in the left, while *h(x)* gets close to 2 when x gets close to a on the right. From the uniqueness property of a limit hence there is no single number L such that *h(x)* gets close to L when x gets close to *a,* we deduct that *h(2)* has no limit.

Finally, let us illustrate function $f(x) = \sin\frac{1}{x}$ ($x \neq 0$). Here (see **Figure 5**) as x gets close to a=0, the value *f(x)* oscillates rapidly. Even if we look only at one side of a, it is clear that there is no number L toward which the value *f(x)* tends. Hence *f* has no limit at 0.

Figure 5 ***f(x)=sin(1/x) (x≠0)***



To emphasise the strong analogy between **Definitions 4-5** let us fill the **'Table of analogues'** (see p.9). If each entry in the right-hand column is substituted for the corresponding entry on the left, we change Definition 4 into Definition 5. However, more than mechanical process is involved here. Corresponding entries in the table actually have the same meaning. For example $\{s_n\}_{n=1}^{\infty}$ is a function of an integer variable.

Furthermore, all algebra operations hold on both concepts with exceptions when we have indeterminate limits, such as $\frac{0}{0}$; $\frac{\infty}{\infty}$; $0\times\infty$; $\infty-\infty$; 0^0 ; ∞^0 ; 1^∞. Many limits of an indeterminate form can be evaluated by a L'Hospitale formula, which uses methods of differential calculus.

Contrary, the forms $\infty\times\infty=\infty$; $\infty+\infty=\infty$; $-\infty-\infty=\infty$ are evidently determinate, in the sense that, for instance, if $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ then $\lim_{x\to a} f(x)\times\lim_{x\to a} g(x)=0$

Other determinate forms are 0^∞ ; $\sqrt[n]{\infty}$.

Expressions of the form *a/0*, where *a* is a non-zero number or ∞ are undefined, because if *y* is very small, then *a/y* will be very large in size but positive or negative according to the sign of *y* and *a*. Also $0^{-\infty}$ is undefined, because $\lim_{x\to 0+}\frac{1}{\sqrt[2]{x}}=+\infty$,but $\lim_{x\to 0-} x^{-g(x)}=-\infty$ if *g(x)* equals the greatest odd integer $\le\frac{1}{x}$.

. **Table of analogues**

Limit of series	$\{s_n\}_{n=1}^{\infty}$	n	s_n	L	∞	ε	N	$n\ge N$
Limit of function (RV)	*f(x)*	x	*f(x)*	L	a	ε	δ	$0<\lvert x-a\rvert<\delta$

As has been stated one of the most important application of the concept of limit is in conjunction with concept of continuity. Intuitively we can deduce that continuous function at a point has no gaps in this point. Formally, using limit concept, we can express this as

Definition 6 We say that the function *f* of a real variable is continuous at a point *a* if $\lim_{x\to a} f(x)=f(a)$.

Definition 6 really demands that 2 conditions be fulfilled in order that *f* be continuous at *a*. The first condition is that the $\lim_{x\to a} f(x)$ must exist; the second is that this limit be equal to *f(a)*. In particular, if *f(x)* is not defined, then *f* cannot be continuous at *a*. For example, the function *f=sinx/x* (*x=0*) is not defined at *x=0* and hence is not continuous at *x=0* even though its limit exists and equal to 1. we can define f(0)=1, then it becomes continuous at (0) since $\lim_{x\to 0} f(x)=f(0)=1$.For the fact that continuity is defined using limit we can deduct that all algebra operations hold for continuous functions.

Conclusions

People use words ‘infinity’, ‘limit’, ‘continuity’ every day: ‘*government applies a limit to someone’s activity’, ‘someone limits one’s ambitions or aspirations’; ‘one’s waiting time on the NHS lasts for an infinite period’, ‘someone may express love as being infinite’; ‘one cannot complete refurbishment because the production of the wallpaper he/she needs is discontinued’, etc.*

We face those concepts everywhere. Musical harmonies have pure mathematical structures and obey rules of harmonic series, which have an explicit relation with limit. The complexity of the concept made it one of the most important in philosophy. Theology appeals to it in the most important doctrines. On the other hand, scientists use these common words to define fundamental concepts of mathematical knowledge. The concept of limit led to differential and integral calculus and modern methods of approximation, which has an infinite variety of applications in modern physics, astronomy, chemistry, engineering, and biology. However, as we emphasised throughout the essay the concept of limit is still beyond understanding in a number of cases; the use of the Cauchy definition solved old problems, there exist undefined or indeterminate forms of limit, that are still unsolved.

Just a final remark: re-phrasing Ludwig Witgenstein ‘ the limits of my words mean the limits of my world’ ([3], p.826). We strongly believe that although we are still limited by our English, the presented discussion delivers and clarifies our view on the most important and conceivably the most difficult concept in mathematics.

Appendix 1

Achilles and the tortoise

We provide calculations supporting the Paradox based on the assumptions:

- tortoise speed- 3m/sec (constant)
- Achilles speed- 6 m/sec (constant)
- head start given to tortoise- 3 meters
- first step:

distance run by Achilles: 3m time spent: $\frac{3m}{6\,m/sec} = 0.5\,sec$

by tortoise: 3m/sec x 0.5 sec=1.5 m

- second step:

distance run by Achilles:1.5m time spent: $\frac{1.5m}{6\,m/sec} = 0.25\,sec$;

by tortoise:3m/sec x 0.25sec=0.75 m

- third step:

distance run by Achilles:0.75m time spent: $\frac{0.75\,m}{6\,m/sec} = 0.125\,sec$;

by tortoise:3m/sec x 0.125sec=0.375 m, etc.

Summing up all these infinite number of interval leads to the infinite geometric sequence with the first term a=½ and the ratio r=½. The definition of the concept of limit following development of the real analysis resulted in the formula of convergence of the geometric series with –1< r <1, which turned the sum of infinite number of terms into a finite figure. Thus,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = \frac{1/2}{1-1/2} = 1\,sec \ .$$

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* Illustrations composed of 'Clip Art-on-line' resources.