Calculus III, 2011: Coursework 4

Will Sutherland.

The deadline is 4PM on Thursday Oct 27th. Hand in **Question 4 only** *to the green box on the basement floor of the Maths building.*

1. Calculate $\nabla \cdot \mathbf{F}$ (i.e. div $\mathbf{F}$) and $\nabla \times \mathbf{F}$ (i.e. curl $\mathbf{F}$) for each of the following vector fields.

(a) $\mathbf{F} = (3x + y)\mathbf{i} + (2x + y)\mathbf{j} + (4z + 3y)\mathbf{k}$

(b) $\mathbf{F} = x^3 \ln z\, \mathbf{i} + y^2 \sin z\, \mathbf{j} + xz^2\, \mathbf{k}$

2. (a) Prove that $\nabla \cdot \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = \mathbf{0}$, where $\mathbf{r}$ is the usual position vector.

(b) If $\mathbf{a}$ is a constant vector, show that $(\mathbf{a} \cdot \nabla)\mathbf{r} = \mathbf{a}$, and $(\mathbf{a} \cdot \nabla)r = \mathbf{a}.\mathbf{r}/r$

3. If U is a scalar field and $\mathbf{F}$ is a vector field, prove from the definitions that

$$\nabla \cdot (U\mathbf{F}) = U(\nabla \cdot \mathbf{F}) + (\nabla U) \cdot \mathbf{F}$$

(this is Equation 3.4 of the lecture notes).

4. **Hand-in question:** Using the identities for differentiation of products,

(a) Prove that if two vector fields $\mathbf{A}$ and $\mathbf{B}$ are irrotational, then $\mathbf{A} \times \mathbf{B}$ is solenoidal. [2]

(b) Prove that for any function $f(r)$, $f(r)\mathbf{r}$ is irrotational. [3]

(c) Let U and V be any two scalar fields. Prove using suitable identities that the Laplacian of their product is given by

$$\nabla^2(UV) = U\nabla^2 V + 2(\nabla U) \cdot (\nabla V) + V\nabla^2 U$$

Hint: start from the definition $\nabla^2 f \equiv \nabla \cdot (\nabla f)$ (Eq. 3.10 of notes) [5]

5. Evaluate the divergence and curl of $\mathbf{a}(\mathbf{r} \cdot \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors. (Hint: you may find it useful to work out $\nabla(\mathbf{b} \cdot \mathbf{r})$ first; see also results from Question 2.)