Calculus III, 2009: Coursework 2
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The deadline is 4PM on Thursday Oct 13th. Hand in your solution for **Question 4 only** *to the green box on the basement floor of the Maths building.*

1. Given a plane defined by $2x - 2y + z = 12$, find:

(a) The unit vector normal to the plane ;

(b) The point in the plane closest to the origin.

(c) The perpendicular distance from the point $(6, -6, 3)$ to the plane.

Answers:
(a) The given equation rewritten in vector form is $\mathbf{r} \cdot \mathbf{a} = 12$ where $\mathbf{a} = (2, -2, 1)$; so the unit vector normal to the plane is $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}| = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$.

(b) Dividing the above equation for the plane by $a = 3$, we have $\mathbf{r} \cdot \hat{\mathbf{a}} = 4$, so $|\mathbf{r}|\, 1 \cos\theta$ $= 4$, where θ is the angle between $\mathbf{r}$ and $\hat{\mathbf{a}}$. The smallest possible r will occur when $\cos\theta = 1$, so $\mathbf{r}$ is parallel to $\hat{\mathbf{a}}$. Thus the distance of the plane from the origin is 4, and the closest point to the origin is $\mathbf{r} = 4\,\hat{\mathbf{a}} = (\frac{8}{3}, -\frac{8}{3}, \frac{4}{3})$.

(c) Let $\mathbf{p} = (6, -6, 3)$ be the position vector of the given point. The line through $\mathbf{p}$ perpendicular to the plane is $\mathbf{r} = \mathbf{p} + t\,\hat{\mathbf{a}}$ (for any real t), where t is the parameter. This line will cross the plane when $(\mathbf{p} + t\hat{\mathbf{a}}).\hat{\mathbf{a}} = 4$. Evaluating $\mathbf{p}.\hat{\mathbf{a}} = 9$, so the solution is $t = -5$. So the perpendicular distance of $\mathbf{p}$ from given plane is 5 (and the closest point to P in given plane is $\mathbf{p} - 5\hat{\mathbf{a}}$).

2. Find the gradient vector (∇f) for $f(x, y, z) = 2x^3y + 3(x^2 - y^2) + z$.

Answer: From the definition of ∇f, taking the partial derivatives of f with respect to x, y, z, we get $\nabla f = (6x^2y + 6x)\mathbf{i} + (2x^3 - 6y)\mathbf{j} + 1\mathbf{k}$.

3. Consider a sphere of radius 9 centred at the origin.

(a) Express the surface of the sphere in the form $f(x, y, z) = 0$.

(b) Find a plane tangent to the sphere at the point $\mathbf{r}_1 = (7, 4, -4)$.

(c) Find the position vector of points on a line through $\mathbf{r}_1$ and normal to the sphere.

(d) Find the position vector of points on the line through $\mathbf{r}_1$ and $\mathbf{r}_2 = (4, 7, 4)$.

Answer: (a) $x^2 + y^2 + z^2 - 81 = 0$, or

$$\frac{x^2}{9^2} + \frac{y^2}{9^2} + \frac{z^2}{9^2} - 1 = 0 \tag{1}$$

(b) Letting $f = x^2 + y^2 + z^2 - 81$, we have $\nabla f = (2x, 2y, 2z)$. At the given point $\mathbf{r}_1$, we have $\nabla f = (14, 8, -8)$, so the equation for the tangent plane is $(\mathbf{r} - \mathbf{r}_1) \cdot (14, 8, -8) = 0$ so $14(x - 7) + 8(y - 4) - 8(z + 4) = 0$, or $7x + 4y - 4z = 81$.

(c) The normal line is $\mathbf{r} = \mathbf{r}_1 + t(14, 8, -8)$. Note choosing $t = -\frac{1}{2}$ gives $\mathbf{r} = \mathbf{0}$, as expected for the special case of a sphere where all normal lines go through the centre of the sphere. In this case our line can be simplified to $\mathbf{r} = u(7, 4, -4)$ for any u, or eliminate u to get $y = \frac{4}{7}x, z = -\frac{4}{7}x$ or similar.

(d) This line is $\mathbf{r} = \mathbf{r}_1 + t(\mathbf{r}_2 - \mathbf{r}_1)$ for any real t. (Note, this is just a straight line through 2 given points, and doesn't depend on f or ∇f). This becomes $x = 7 - 3t,\ y = 4 + 3t,\ z = -4 + 8t$; if desired, we can rearrange the x equation to $t = (7 - x)/3$, and plug that into the y, z equations to get y and z in terms of x.

4. (*) Hand-in question:

(a) Calculate the volume of a parallelepiped (the 3D version of a parallelogram) where 3 edges are given by the vectors $2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. [5]

(b) Find equations for the (i) tangent plane and (ii) normal line, to the surface $3x^2 - y^2 + 2z^2 = 1$, at the point $P_0 = (1, 2, -1)$. [5]

Answer: (a) Call the three given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Then volume of the parallelepiped is the absolute value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$; evaluating, we get $\mathbf{b} \times \mathbf{c} = 1\mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$, and so the scalar triple product is 15.

(b) Remember that for ANY surface $f(x, y, z) = c$, for c any constant, the tangent plane and normal line at a point are respectively perpendicular and parallel to ∇f at that point (as long as $\nabla f \neq \mathbf{0}$ at the point). For the given function we get $\nabla f = (6x, -2y, 4z)$, so at the given point $\nabla f = (6, -4, -4)$. Hence the tangent plane is $(\mathbf{r} - \mathbf{P}_0) \cdot (6, -4, -4) = 0$, or writing out components, $6(x - 1) - 4(y - 2) - 4(z + 1) = 0$. This simplifies to $3x - 2y - 2z = 1$.

The normal line is $\mathbf{r} = \mathbf{P}_0 + t(\nabla f)_{P_0}$, for any t, so $\mathbf{r} = (1, 2, -1) + t(6, -4, -4)$, or $x = 1 + 6t, y = 2 - 4t, z = -1 - 4t$. If we want y, z in terms of x, rearrange the x equation to $t = (x - 1)/6$, and get $y = \frac{8}{3} - \frac{2}{3}x$ and $z = -\frac{1}{3} - \frac{2}{3}x$.

5. Find the directional derivative of the scalar field $V = 4y^2 z + xyz^2$ in the direction of the vector $(2, -1, -1)$ at the point $P = (1, -2, -1)$. [Hint: write down the unit vector $\mathbf{n}$ which has the same direction as the given vector, and evaluate $\mathbf{n} \cdot \nabla V$ at the point P.]

Answer:
From definition of grad, we have $\nabla V = (0 + yz^2, 8yz + xz^2, 4y^2 + 2xyz)$. At point P, this is $(-2, 17, 20)$. Our given vector is $(2, -1, -1)$ so the unit vector parallel to this is $\mathbf{n} = (2, -1, -1)/\sqrt{6}$. Hence at P, the directional derivative is $\mathbf{n} \cdot \nabla V = -41/\sqrt{6}$.

6. Prove that $\nabla(r^n) = nr^{n-2}\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the usual position vector and $r = |\mathbf{r}| = (x^2 + y^2 + z^2)^{1/2}$.

Answer
To find the gradient we need $\partial(r^n)/\partial x$ and so on, so we need the chain rule. First, we have $r = \sqrt{x^2+y^2+z^2}$ so $\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r}$. Then, by the chain rule, for $f = r^n$,

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-2}x$$

The results for y and z are similar, so we get

$$\begin{aligned}\nabla f &= nr^{n-2}x\,\mathbf{i} + nr^{n-2}y\,\mathbf{j} + nr^{n-2}z\,\mathbf{k}\\ &= nr^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\\ &= nr^{n-2}\mathbf{r}.\end{aligned}$$

(Note it can also be done from $r^n = (x^2+y^2+z^2)^{n/2}$, and the chain rule on that).

The answer can also be written as $nr^{n-1}\,\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector parallel to $\mathbf{r}$ and $\mathbf{r} \equiv r\,\hat{\mathbf{r}}$.

(**Hint:** remember this formula, it is likely to come up again in later parts of the course).