A Cyclic Property of Cyclic Polygons Revisited (Again)

CAPTAINBLACK[1]

The following paper is a result of a question posed by user Natasha on Math Help Forum in November 2005 (Reference 1). Given the date of posting, and that of publication of Reference 2 we may assume that Natasha's homework was based on that reference. What follows is a preliminary version of what was eventually published under the name of a good friend of the present author as Reference 4. The delay between the posting of this on MHF and appearing in print was primarily because it took at least a year to trace the origin of the problem, and the usual delays of publication.

In Reference 2 Chris Pritchard presents the result that the sum of every other angle of a convex cyclic 2n-gon is 2(n-1) right angles. In this note I present an alternative proof of this result by showing that such an angle sum is invariant as the vertices are moved subject to certain constraints, and since subject to these constraints a convex 2n-gon can be transformed into a regular 2n-gon the value of such an angle sum is the same as that of the 2n-gon.

The key observation in my alternative demonstration of this result is that for any three neighbouring vertices of the polygon the middle vertex forms an angle in the sector between the other two vertices. So the interior angle at this vertex is independent of the position of the vertex on the sector, as the angle is the same for all positions, this is the content of Proposition 21 of Book 3 of The Elements of Euclid, and is also Proposition 4.14 in reference 3. So any transformation of the polygon where a vertex moves to a new position on the sector between its neighbours leaves the angle sum of interior angles for the set of alternate vertices including the moved vertex unchanged.

Perhaps surprisingly moving a vertex on the sector between its neighbours also leaves the angle sum of the other set of alternate vertices unchanged. This is because while the angles at the neighbouring vertices do change their sum remains unchanged. This is because the angle at the moved vertex is unchanged and so the sum of the other angles in the triangle formed by the three vertices must remain constant to maintain the triangles angle sum at 180°. This is illustrated in Figure 1.

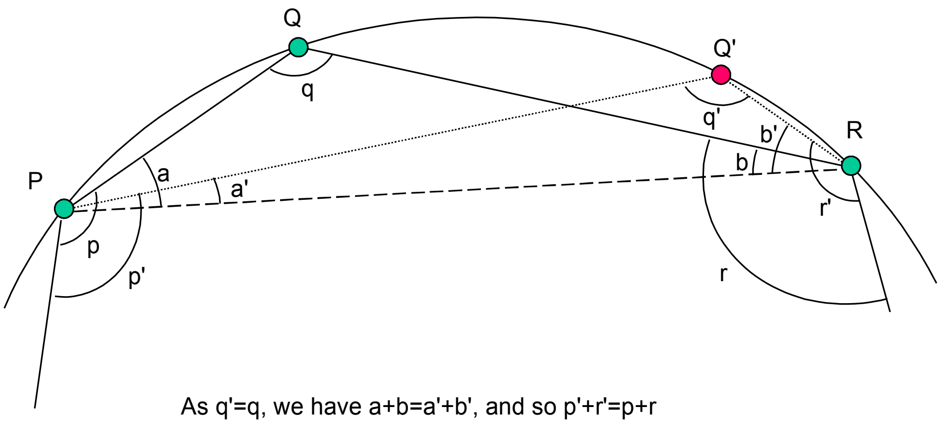

Figure 1

So we may conclude that as a vertex is moved along the arc connecting its neighbours that both of the sums of alternate internal angles remain unchanged.

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Now we are in a position to give a neat demonstration of Chris Pritchard's result. Choose a vertex of our 2n-gon to use as a reference vertex. Now move the other vertices in turn so that they retain their order and all end closer to the reference vertex than a vertex in a regular 2n-gon would be. This can be done in such a way that we never change the two alternate angle sums of the polygons. Now we can move the vertices in turn to the positions of the vertices of a regular 2n-gon, again without changing the alternate angle sums. Figure 2 shows the stages in this process for a hexagon.

Three stages in the transformation process to transform a non-regular hexagon into a regular hexagon while preserving the interior angle sum for alternate vertices

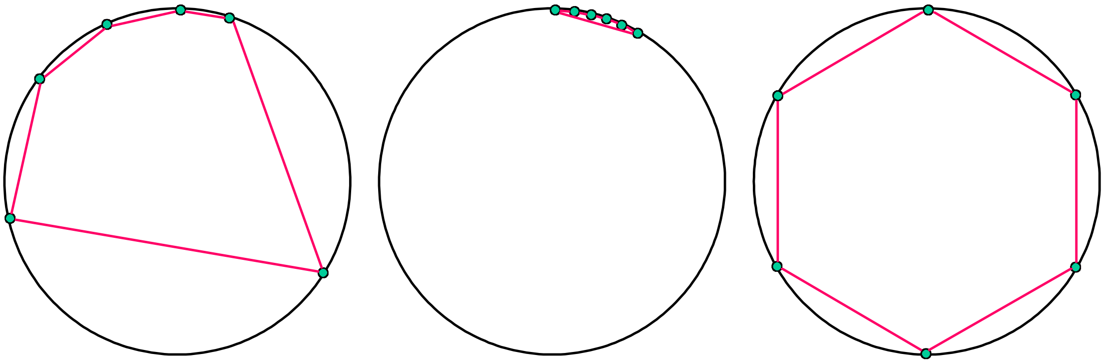

Figure 2

Hence the two alternate angle sums of our 2n-gon are the same as those of a regular 2n-gon, which are easily demonstrated to be (n-1)x180°.

References

1. Natasha, Cyclic Hexagon, Math help Forum 2005, http://www.mathhelpforum/math-help/advanced-geometry/1244-cyclic-hexagon.html
2. Pritchard, C. (2005). A Cyclical Property of Cyclic Polygons. Mathematics in School, November, 34(5), p.14.
3. Silvester, John R. (2001). Geometry Ancient and Modern, OUP
4. Xxxxxx (2007). A Cyclical Property of Cyclic Polygons Revisited. Mathematics in School, September 2007.