*University of Edinburgh*

# DYNAMICS 3 – ASSIGNMENT

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**19/03/2008**

# 1. INTRODUCTION

A laboratory uses a three cylinder inline engine as its emergency power generator (figure 1). This engine is mounted on four supports (two hidden behind the two visible supports in figure 1) on a concrete floor.

Te data that is known for this assignment is the total mass (M = 100kg), effective moving mass of each cylinder (m = 400g), the length of the crank shafts (r = 6cm), the length of the con rods (l = 20cm) and the steady state speed of the engine (4900 rpm).

The aims of this assignment are as follows:

- to devise a balancing scheme for the engine and find the angles at which the cranks should be attached to minimise unbalanced forces and moments
- to evaluate the residual forces acting on the supports as a function of the rotational speed of the engine and find the forces present during steady state operation
- to devise a method of reducing force transmission through the supports to the following specifications:
  - the maximum force transmitted during steady state operation through any support must not exceed 100N
  - the maximum force amplitude transmitted during run up through any of the supports must not exceed 400N
  - The total mass of the engine plus mounting should not increase by more than 10kg
  - The vertical displacement of the engine under its own weight and dynamic vibration amplitude should be kept as small as possible
- to find a reason why the method of placing the cranks at angles of $2\pi/n$ (where n = number of cylinders) is not implemented for four cylinder engines

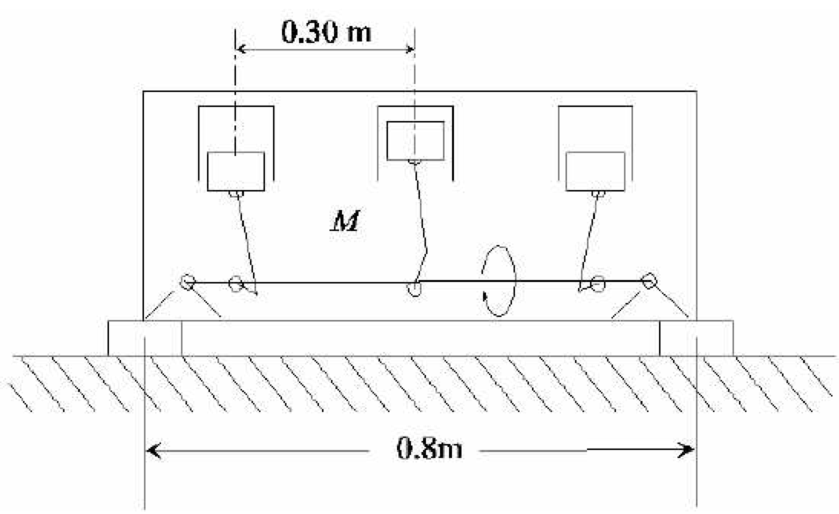



Figure 1

# 2. DESIGN SOLUTIONS

## 2.1 Balancing Scheme

It is necessary to arrange the cranks at such angles to minimise internal unbalanced moments and forces in any type of engine. The most logical method to use is to arrange the cranks at equal angles from each other and as there are three cylinders the angle between each one would be $2\pi/3 = 120°$. Another method, which is utilised in four cylinder engines, is to arrange the cranks at an angle of $\pi$ from each other in such an order as to produce a smooth firing sequence. Although this method is utilised in many three cylinder engines, it results in a missing firing sequence as it is the equivalent of the four cylinder engine expect without the fourth piston. I will look at both of these balancing schemes in order to see the benefits and drawbacks of each and choose a suitable one to carry forward.

The first step is to find the equations necessary to equate the unbalanced forces and moments. This begins by looking at the piston schematic below and forming the following equations:

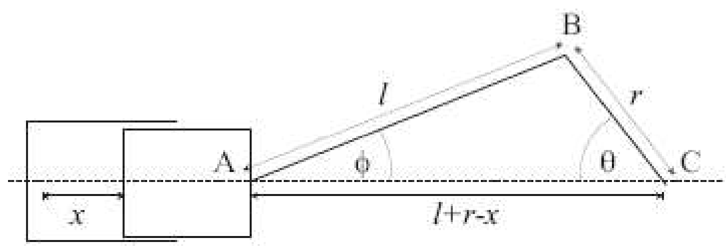



Figure 2

$AB\cos\phi + BC\cos\theta = AC \Rightarrow l\cos\phi + r\cos\theta = l + r - x$ - Equation 1

Sine law: $l\sin\phi = r\sin\theta$

And we can eliminate $\phi$ as $\cos\phi = \sqrt{1-\sin^2\phi} = \sqrt{1-\frac{r^2\sin^2\theta}{l^2}}$

$\therefore x = r\left(1-\cos\theta + \frac{\lambda\sin^2\theta}{2}\right)$ - Equation 2, where $\lambda$ is small and equals r/l

Differentiating $x$ with respect to t where $\theta = \Omega t$ and $\sin(\Omega t)\cos(\Omega t) = \frac{1}{2}\sin(2\Omega t)$ :

$\dot{x} = r\Omega\left(\sin(\Omega t) + \lambda\sin(\Omega t)\cos(\Omega t)\right) = r\Omega\left[\sin(\Omega t) + \frac{\lambda}{2}\sin(2\Omega t)\right]$ - Equation 3

$\ddot{x} = r\Omega^2\left(\cos(\Omega t) + \lambda\cos(2\Omega t)\right)$ - Equation 4

$F = m\ddot{x} = mr\,\Omega^2\left[\cos(\Omega t) + \lambda\cos(2\Omega t)\right]$ - Equation 5

Where the first cosine term is the primary force and the second cosine term at twice the frequency is the secondary force.

Generally: $F_i = mr\,\Omega^2\left[\cos(\Omega t + \alpha_i) + \lambda\cos 2(\Omega t + \alpha_i)\right]$ - Equation 6

Therefore,

$F_p = mr\,\Omega^2\left[\cos\Omega t\sum\cos\alpha_i - \sin\Omega t\sum\sin\alpha_i\right]$ - Equation 7

$F_s = mr\,\Omega^2\lambda\left[\cos 2\Omega t\sum\cos 2\alpha_i - \sin 2\Omega t\sum\sin 2\alpha_i\right]$ - Equation 8

We also want to find the unbalanced moments, which can be found by locating the centre of gravity (O) in the middle of the engine and each crank is $z_i$ away from O:

$M_p = mr\,\Omega^2\left[\cos\Omega t\sum z_i\cos\alpha_i - \sin\Omega t\sum z_i\sin\alpha_i\right]$ - Equation 9

$M_s = mr\,\Omega^2\lambda\left[\cos 2\Omega t\sum z_i\cos 2\alpha_i - \sin 2\Omega t\sum z_i\sin 2\alpha_i\right]$ - Equation 10

We can now construct the following tables (table 1 to table 4) which will allow us to find if there are any unbalanced forces or moments in the engine for both of the balancing schemes mentioned earlier.

| **Crank** | **$\alpha_i$** | **$z_i$** | **$\sin\alpha_i$** | **$\cos\alpha_i$** | **$\sin 2\alpha_i$** | **$\cos 2\alpha_i$** |
|---|---|---|---|---|---|---|
| **1** | $2\pi/3$ | -0.3 | $\sqrt{3}/2$ | $-1/2$ | $-\sqrt{3}/2$ | $-1/2$ |
| **2** | 0 | 0 | 0 | 1 | 0 | 1 |
| **3** | $4\pi/3$ | 0.3 | $-\sqrt{3}/2$ | $-1/2$ | $\sqrt{3}/2$ | $-1/2$ |
| **Total** | | | 0 | 0 | 0 | 0 |

Table 1 – Unbalanced forces when arranged at angles of 120° from each other

| **Crank** | **$\alpha_i$** | **$z_i$** | **$\sin\alpha_i$** | **$\cos\alpha_i$** | **$\sin 2\alpha_i$** | **$\cos 2\alpha_i$** |
|---|---|---|---|---|---|---|
| **1** | 0 | -0.3 | 0 | 1 | 0 | 1 |
| **2** | π | 0 | 0 | -1 | 0 | 1 |
| **3** | 0 | 0.3 | 0 | 1 | 0 | 1 |
| **Total** | | | 0 | 1 | 0 | 3 |

Table 2 – Unbalanced forces when arranged at angles of 180° from each other

| Crank | $\alpha_i$ | $z_i$ | $z_i \sin\alpha_i$ | $z_i \cos\alpha_i$ | $z_i \sin 2\alpha_i$ | $z_i \cos 2\alpha_i$ |
|---|---|---|---|---|---|---|
| **1** | $2\pi/3$ | -0.3 | $-0.3\times\sqrt{3}/2$ | $0.3/2$ | $-\sqrt{3}/2$ | $0.3/2$ |
| **2** | 0 | 0 | 0 | 0 | 0 | 0 |
| **3** | $4\pi/3$ | 0.3 | $0.3\times-\sqrt{3}/2$ | $0.3/2$ | $\sqrt{3}/2$ | $-0.3/2$ |
| **Total** | | | $-0.3\sqrt{3}$ | 0 | $0.3\sqrt{3}$ | 0 |

Table 3 – Unbalanced moments when arranged at angles of 120° from each other

| Crank | $\alpha_i$ | $z_i$ | $z_i \sin\alpha_i$ | $z_i \cos\alpha_i$ | $z_i \sin 2\alpha_i$ | $z_i \cos 2\alpha_i$ |
|---|---|---|---|---|---|---|
| **1** | 0 | -0.3 | 0 | -0.3 | 0 | -0.3 |
| **2** | π | 0 | 0 | 0 | 0 | 0 |
| **3** | 0 | 0.3 | 0 | 0.3 | 0 | 0.3 |
| **Total** | | | 0 | 0 | 0 | 0 |

Table 4 – Unbalanced moments when arranged at angles of 180° from each other

A can be seen from tables 1 through 4 above, when placing the cranks at angles of 120° there are no unbalanced forces but a primary and secondary moment are present while the opposite case is true when the cranks are at an angle of 180°. Either case is valid to carry forward but as only one can be chosen the case of cranks placed at 120° will be investigated in the later stages of this assignment.

## 2.2 Residual Forces

The forces present at the supports can be evaluated by using the unbalanced moments in table 3. Therefore, equations 9 and 10 become:

$$M_p = 0.3*\sqrt{3}*mr\Omega^2(\sin\Omega t)$$

$$\therefore F_p = 0.3*\sqrt{3}*\frac{1}{0.4}*mr\Omega^2(\sin\Omega t) \qquad \text{- Equation 11}$$

$$M_s = 0.3*\sqrt{3}*mr\Omega^2\lambda(-\sin 2\Omega t)$$

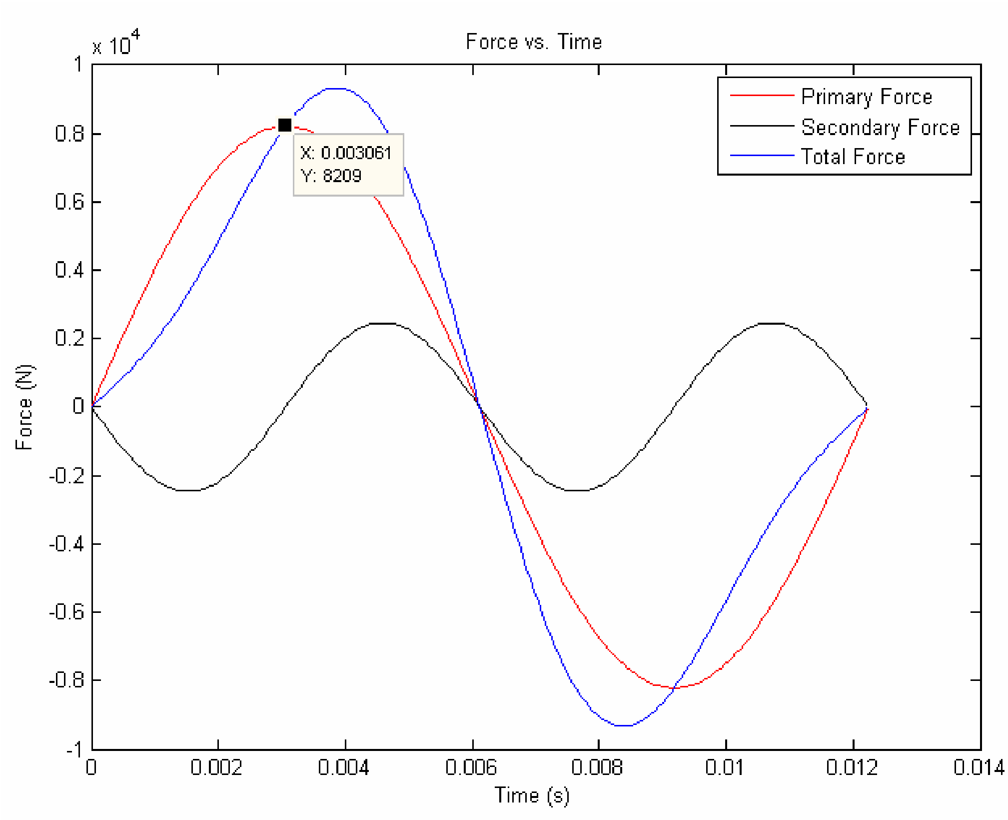
$$\therefore F_s = 0.3*\sqrt{3}*\frac{1}{0.4}*mr\Omega^2\lambda(-\sin 2\Omega t) \qquad \text{- Equation 12}$$

The maximum forces occur when $\sin\Omega t = 1$ and $\sin 2\Omega t = -1$, therefore,

$$F_p = 0.3*\sqrt{3}*\frac{1}{0.4}*mr\Omega^2 = 0.3*\sqrt{3}*\frac{1}{0.4}*0.4*0.06*513.13^2 = 8209\ N$$

$$F_s = 0.3 * \sqrt{3} * \frac{1}{0.4} * mr\,\Omega^2 \lambda = 0.3 * \sqrt{3} * \frac{1}{0.4} * 0.4 * 0.06 * 513.13^2 * \frac{0.06}{0.2} = 2463 \quad N$$

However, these maximums will not occur at the same time due to their different frequencies, therefore, it is necessary to look at the forces created throughout the cycle and combine the primary and secondary forces to find the maximum as can be seen in graph 1. The maximum force acting on the supports is therefore 9329N.




Graph 1 – Forces during steady state operation when no vibration insulation is used

## 2.3 Force Transmission Reduction

There are numerous methods of reducing the forces transmitted to the mounting due to the unbalanced moments and forces of engines. Common methods include an extra shaft, out of balance rotors and extra masses to act as vibration absorbers. However, I will not look at any of these methods and rather look at the simple case of using some form of vibration insulation that is a combination of a spring and place them where the supports are currently located.

The first step was to look at the system and draw the associated FBD (figure 3). Once this was done terms for the natural frequency ($\omega_0$), the damping ratio ($\delta$) and ( $x_0$ ) can be found.

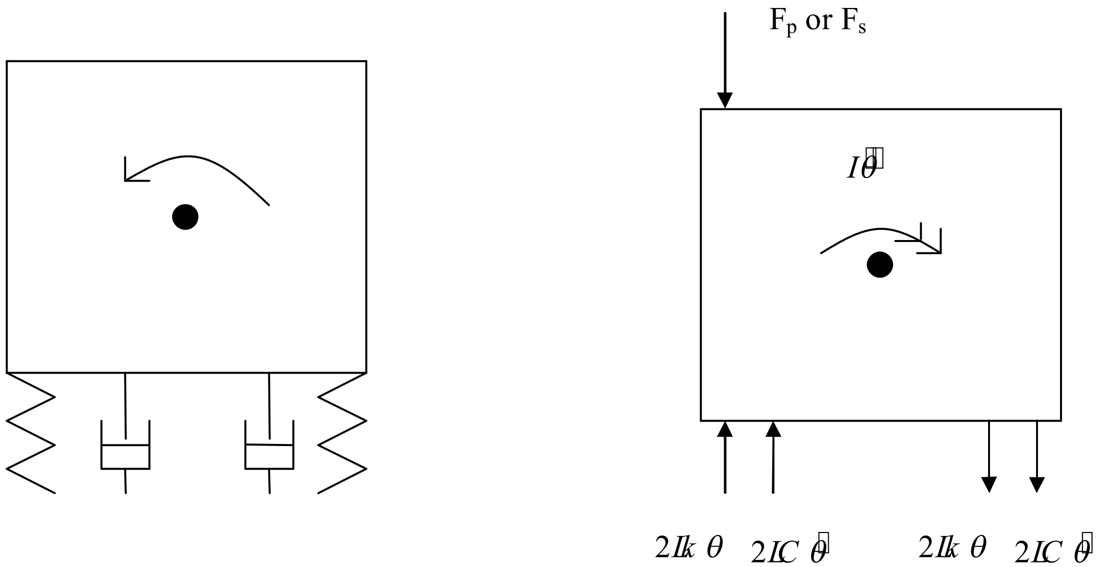



Figure 3 – System sketch and FBD

$$I\ddot{\theta} + 4L^2c\dot{\theta} + 4L^2k\theta = FL \Rightarrow \ddot{\theta} + \frac{4L^2c}{I}\dot{\theta} + \frac{4L^2k}{I}\theta = \frac{FL}{I}$$ - Equation 13

Canonical Form: $\ddot{\theta} + 2\omega_0\delta\dot{\theta} + \omega_0{}^2\theta = \omega_0{}^2 x_0$

$$\omega_0{}^2 = \frac{4L^2k}{I} \Rightarrow \omega_0 = \sqrt{\frac{4L^2k}{I}}$$ - Equation 14, where $I = \frac{1}{12}M(h^2 + w^2)$

$$2\omega_0\delta = \frac{4L^2c}{I} \Rightarrow \delta = \frac{4L^2c}{2\omega_0 I}$$ - Equation 15

$$\omega_0{}^2 x_0 = \frac{FL}{I} \Rightarrow x_0 = \frac{FL}{I\omega_0{}^2}$$ - Equation 16

We now have all the terms necessary to find the dynamic vibration amplitude, transmission ratio and the transmitted forces:

$$A = \frac{x_0}{\sqrt{\left(1 - \frac{\Omega^2}{\omega_0^{\ 2}}\right)^2 + 4\delta^2 \frac{\Omega^2}{\omega_0^{\ 2}}}} \qquad \text{- Equation 17}$$

$$TR = \sqrt{\frac{1 + 4\delta^2 \Omega^2 / \omega_0^{\ 2}}{\left(1 - \frac{\Omega^2}{\omega_0^{\ 2}}\right)^2 + 4\delta^2 \frac{\Omega^2}{\omega_0^{\ 2}}}} \qquad \text{- Equation 18}$$

$$f_{s,0} = f_0 \times TR \qquad \text{- Equation 19}$$

As there are two different forces acting on the system (primary and secondary), there are two different $x_0$ 's, amplitudes and transmission ratios, one with normal speed Ω (primary) and the other with twice the speed 2Ω (secondary).

As the system will experience static sag due to the weight of the engine on the supports it is necessary to use a spring with a high enough stiffness to reduce the sag to an acceptable level. The following equation can be used:

$$F = kx \Rightarrow k = \frac{Mg}{4x} \qquad \text{- Equation 20}$$

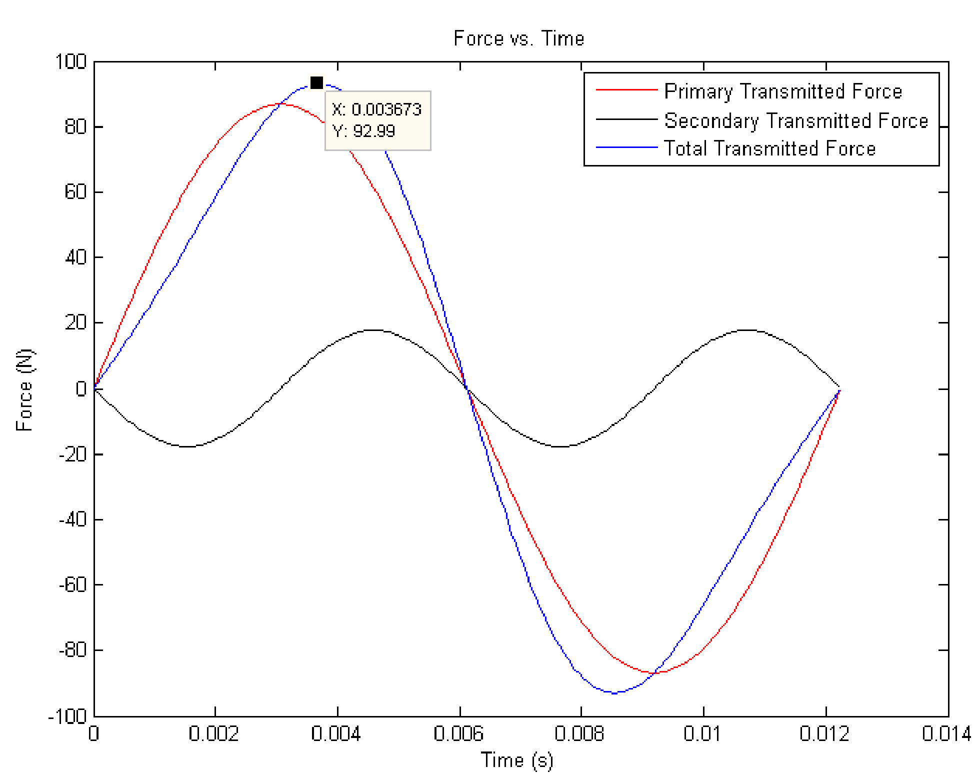
The next step is to state a displacement that would be considered suitable and find the most appropriate stiffness to use. I believe that value of 2.5cm would be acceptable, which results in the following overall value of k:

$$k = \frac{100 * 9.81}{4 * 0.025} \approx 10000 \quad N/m$$

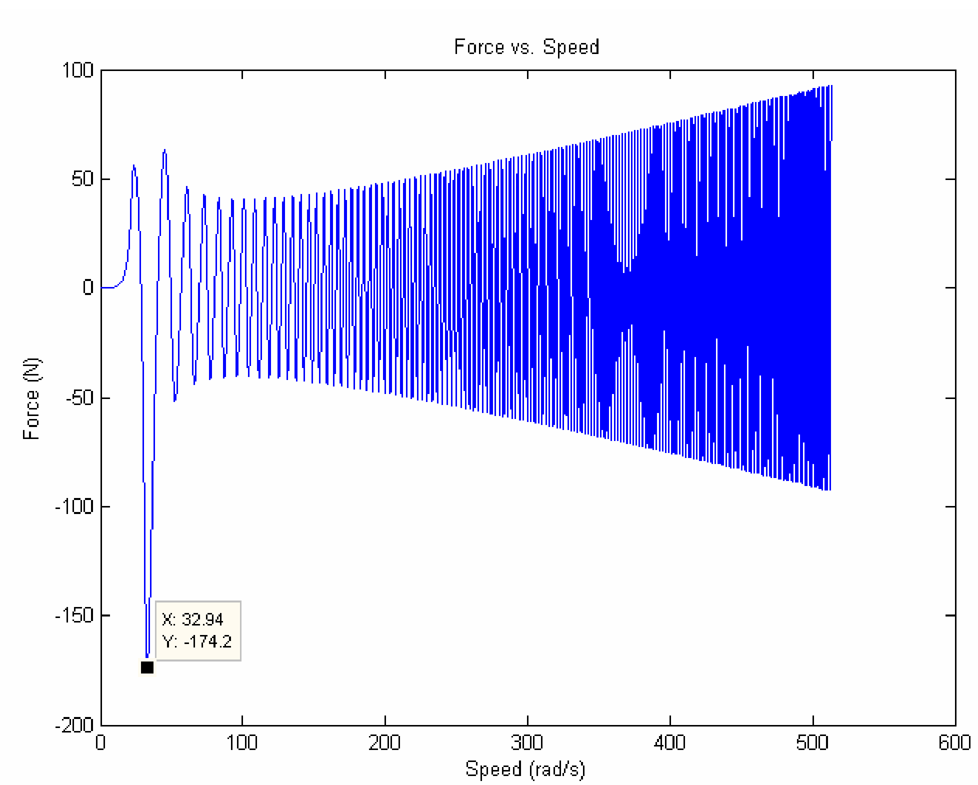
The stiffness of the individual springs at each support is therefore 2500Nm. We can use this value of k to find the natural frequency of the engine from equation 14, $\omega_0$ = 31.74 rad/s.

We now have all the values needed to find the required forces except from the damping coefficient and damping ratio. These were found by placing the calculations into MATLAB and altering the damping coefficient, c, until the values for the forces met the required specifications. The most efficient value used for the damping coefficient was found to be c = 50 Ns/m and this resulted in a damping ratio of δ = 0.079.

The results of the MATLAB calculations (appendix 1) can be seen in graphs 2 and 3. Graph 2 shows the primary, secondary and combined transmitted forces once the vibration insulation had been implemented into the system and it is clear that the maximum force is below the required 100N as the value is 93N.




Graph 2 – Transmitted forces during steady state operation with damping system implemented




Graph 3 – Forces during run-up of engine with damping system implemented

Graph 3 shows the varying forces during the run-up of the engine. The maximum value was found to be 174.2 N at an angular speed of 32.94 rad/s. This is what was expected as the maximum force will occur at the point of the natural frequency.

The maximum dynamic vibration amplitude at steady state was also calculated in the MATLAB scripts and was found to be 0.0021m, which is well within the limits of an acceptable value.

The following assumptions were made during this assignment: the transmission ratio equation is the same for rotating systems as single plane vibrating systems; the unbalanced forces found from the unbalanced moments can be treated as external oscillating forces acting on the system with differing frequencies

## 2.4 Advantages and Disadvantages

### 2.4.1 Advantages

- cheap and easy to implement
- no added parts so no affect on mechanical complexity
- the mass is not affected
- Easily replaced/repaired

### 2.4.2 Disadvantages

- may need maintained on a frequent basis
- may be hard to source vibration insulation that has the required specifications
- does not eliminate the unbalanced moments within the engine

# APPENDIX 1

```
M=100; %total mass (kg)
m=0.4; %effective moving mass of piston (kg)
r=0.06; %length of crank shafts (m)
l=0.2; %length of con rods (m)
h=0.35; %estimated height of engine (m)
W=0.8; %width of engine (m)
L=0.4; %distance from COG to supports (m)
omega=4900*((2*pi)/60); %speed of engine (rad/s)
f=omega/(2*pi); %frequency of engine (Hz)
t=1/f; %period (s)
I=(1/12)*M*((h^2)+(W^2)); %second moment of area of engine (kgm2)
k=10000; %spring stiffness (N/m)
c=50; %damping coefficient (Ns/m)
w=sqrt((4*(L^2)*k)/I); %natural frequency (rad/s)
delta=((4*(L^2)*c)/(2*w*I)); %damping ratio
xp=(L*sqrt(3)*0.3*(1/0.4)*m*r*omega^2)/(I*w^2); %x0 for primary force
xs=(L*sqrt(3)*0.3*(1/0.4)*m*r*omega^2*(r/l))/(I*w^2); %x0 for secondary
force
Ap=xp/sqrt(((1-(omega^2/w^2))^2)+((4*delta^2)*(omega^2/w^2))); %dynamic
vibration amplitude under primary force (m)
As=xs/sqrt(((1-((2*omega)^2/w^2))^2)+((4*delta^2)*((2*omega)^2)/w^2));
%dynamic vibration amplitude under secondary force (m)
A=As+Ap %total dynamic vibration amplitude (m)

i=1;
for T=(0:t/200:t);
T1(1,i)=T;
Mp(1,i)=m*r*(omega^2)*(sin(omega*T1(1,i))*(0.3)*sqrt(3)); %primary
moment (Nm)
Ms(1,i)=m*r*(r/l)*((omega^2)*(-sin(2*omega*T1(1,i)))*(0.3)*sqrt(3));
%secondary moment (Nm)
Fp(1,i)=Mp(1,i)/0.4; %primary force due to primary moment (N)
Fs(1,i)=Ms(1,i)/0.4; %secondary force due to secondary moment (N)
F_Total=Fp+Fs; %total force (N)
TR1=sqrt((1+(4*delta^2*omega^2)/w^2)/(((1-
(omega^2)/(w^2))^2)+(4*delta^2*(omega^2/w^2)))); %primary transmission
ratio
TR2=sqrt((1+(4*delta^2*2*omega^2)/w^2)/(((1-
(2*omega^2)/(w^2))^2)+(4*delta^2*(2*omega^2/w^2)))); %secondary
transmission ratio
TR_Fp(1,i)=Fp(1,i)*TR1; %primary transmitted force (N)
TR_Fs(1,i)=Fs(1,i)*TR2; %secondary transmitted force (N)
TR_Total=TR_Fp+TR_Fs; %total transmitted force (N)
i=i+1;
end

plot (T1, TR_Fp)
hold on
plot (T1, TR_Fs)
hold on
plot (T1, TR_Total)
```

Script 1

```
M=100; %total mass (kg)
m=0.4; %effective moving mass of piston (kg)
r=0.06; %length of crank shafts (m)
l=0.2; %length of con rods (m)
h=0.35; %estimated height of engine (m)
W=0.8; %width of engine (m)
L=0.4; %distance from COG to supports (m)
omega=4900*((2*pi)/60); %speed of engine (rad/s)
f=omega/(2*pi); %frequency of engine (Hz)
t=1/f; %period (s)
I=(1/12)*M*((h^2)+(W^2)); %second moment of area of engine (kgm2)
k=10000; %spring stiffness (N/m)
c=50; %damping coefficient (Ns/m)
w=sqrt((4*(L^2)*k)/I); %natural frequency (rad/s)
delta=((4*(L^2)*c)/(2*w*I)); %damping ratio

i=1;
for Omega=0:omega/10000:omega;
O(1,i)=Omega;
T1(1,i)=Omega/(omega/2);
Mp(1,i)=m*r*(Omega^2)*(sin(Omega*T1(1,i))*(0.3)*sqrt(3)); %primary
moment (Nm)
Ms(1,i)=m*r*(r/l)*((Omega^2)*(-sin(2*Omega*T1(1,i)))*(0.3)*sqrt(3));
%secondary moment (Nm)
Fp(1,i)=Mp(1,i)/0.4; %primary force due to primary moment (N)
Fs(1,i)=Ms(1,i)/0.4; %secondary force due to secondary moment (N)
F_Total=Fp+Fs; %total force (N)
TR1=sqrt((1+(4*delta^2*Omega^2)/w^2)/(((1-
(Omega^2)/(w^2))^2)+(4*delta^2*(Omega^2/w^2)))); %primary transmission
ratio
TR2=sqrt((1+(4*delta^2*2*Omega^2)/w^2)/(((1-
(2*Omega^2)/(w^2))^2)+(4*delta^2*(2*Omega^2/w^2)))); %secondary
transmission ratio
TR_Fp(1,i)=Fp(1,i)*TR1; %primary transmitted force (N)
TR_Fs(1,i)=Fs(1,i)*TR2; %secondary transmitted force (N)
TR_Total=TR_Fp+TR_Fs; %total transmitted force (N)
i=i+1;
end

plot (O, TR_Total)
```

Script 2