IN1004 Mathematics for Computing

Lecturer: Dr. Peter W.H. Smith
peters@soi.city.ac.uk

1. Set Theory

1.1 Introduction

A set is one of the most fundamental cornerstones of mathematics. It is a *well-defined* collection of objects. These objects are called *elements* and are said to be *members* of the set. Well-defined implies that we are able to determine whether it is the set under scrutiny. Thus we avoid sets based on opinion, e.g. the set of all great football players.

1.2 Notation and Set membership

Capital letters, **A,B,C**… are used to represent sets and lowercase letters are used to represent elements. For a set **A**, we write $\mathbf{x} \in A$ if **x** is an element of **A; y** $\notin$ A indicates that **y** is not a member of **A**.

A set can be designated by listing its elements within *set braces*. For example, if **A** is the set consisting of the first five positive integers, then we write **A = {1,2,3,4,5}.** In this example, **2** $\in$**A** but **6** $\notin$**A**.

Another standard notation for this set is **A = {x | x is an integer and 1** $\leq$ **x** $\leq$ **5}.** The vertical line | within the set may be read as "such that", the symbols **{x |…}** are read "the set of all **x** such that…" The properties following | help us to determine the elements of the set that is being described.

Note: The notation **{x | 1** $\leq$ **x** $\leq$ **5}** is not an adequate description of the set A unless we have agreed in advance that the elements under consideration are integers. When such an agreement is adopted, we say that we are specifying a **universe**, or **universe of discourse** which is usually denoted by $\mathcal{U}$. We then only select elements of $\mathcal{U}$ to form our sets. In this particular problem, if $\mathcal{U}$ denoted the set of all integers, or the set of all positive integers, then **{x | 1** $\leq$ **x** $\leq$ **5}** adequately describes A. If $\mathcal{U}$ is the set of all real numbers, then **{x | 1** $\leq$ **x** $\leq$ **5}** would contain all of the real numbers between 1 and 5 inclusively. If $\mathcal{U}$ consisted of only even integers, then the only members of **{x | 1** $\leq$ **x** $\leq$ **5}** would be 2 and 4.

Examples

For $\mathcal{U}$ **= {1,2,3,…},** the set of positive integers, let

1. $\mathbf{A = \{1,4,9,16,25,36,49,64,81\} = \{x^2 \mid x \in \mathcal{U}, x^2 < 100\} = \{x^2 \mid x \in \mathcal{U} \wedge x^2 < 100\}}$.
2. $\mathbf{B = \{1,4,9,16\} = \{y^2 \mid y \in \mathcal{U}, y^2 < 20\}}$
3. $\mathbf{C = \{2,4,6,8,\ldots\} = \{2k \mid k \in \mathcal{U}}$

1.3 Cardinality of Sets and Subsets

Sets **A** and **B** are examples of *finite sets*, whereas **C** is an *infinite set*. For any finite set **A**, **|A|** denotes the number of elements in **A** and is referred to as the **cardinality**, or **size** of **A**. In the example above **|A| = 9** and **|B| = 4**.

If **C, D** are sets from a universe $\mathcal{U}$, **C** is a subset of **D**, written **C** $\subseteq$ **D**, or **D** $\supseteq$ **C**, if every element of **C** is an element of **D**. Additionally, if **D** contains an element that is not in **C**, then **C** is called a **proper subset** of **D**, denoted **C** $\subset$ **D** or **D** $\supset$ **C**.

For the universe $\mathcal{U}$ **= {1,2,3,4,5},** consider the set **A = {1,2}.** If **B ={x |** $x^2 \in \mathcal{U}$ **}** then the members of **B** are 1,2. Hence **A** and **B** contain the same elements – and no other, thus sets **A** and **B** should be equal. It is also true that **A** $\subseteq$ **B** and **B** $\supseteq$ **A** giving us the following definition of equality.

For a given universe $\mathcal{U}$, the sets **C** and **D** (taken from $\mathcal{U}$) are said to be equal, written **C=D**, when **C** $\subseteq$ **D**, or **D** $\supseteq$ **C**.

We can see that neither order nor repetition is relevant for a set. Consequently, we find that **{1,2,3} = {3,2,1} = {2,2,1,3} = {1,2,1,2,3}.**

We can define the negation of a subset. **A** /$\subseteq$ **B** if there is at least one element x in the universe where x is a member of **A** but **x** is not a member of **B**.

We have introduced four ideas so far:

1. set membership
2. set equality
3. subsets
4. proper subsets

Examples

Let $\mathcal{U}$ **= {1,2,3,4,5,6,x,y,{1,2},{1,2,3},{1,2,3,4}}** – where **x,y** are simply letters of the alphabet. Then | $\mathcal{U}$ | **= 11.**

a. If **A = {1,2,3,4},** then **|A| = 4** and:
   - **i.** **A** $\subseteq \mathcal{U}$
   - **ii.** **A** $\subset \mathcal{U}$
   - **iii.** **A** $\in \mathcal{U}$
   - **iv.** **{A}** $\subseteq \mathcal{U}$
   - **v.** **{A}** $\subset \mathcal{U}$, but
   - **vi.** **{A}** $\notin \mathcal{U}$

b. Let **B = {5,6,x,y,A} = {5,6,x,y,{1,2,3,4}}.** Then **|B| = 5** (not 8). Now
   - **i.** **A** $\in$ **B**
   - **ii.** **{A}** $\subseteq$ **B** and
   - **iii.** **{A}** $\subset$ **B,** but
   - **iv.** **{A}** $\notin$ **B**
   - **v.** **A** /$\subseteq$ **B** (A is not a subset of B)
   - **vi.** **A** $\not\subset$ **B** (A is not a proper subset of B)

1.4 The Empty Set and Powersets

The *null set*, or *empty set*, is the (unique) set containing no elements. It is denoted by $\varnothing$ or **{}**. Note that $|\varnothing|$ = **0** but **{0}** $\neq \varnothing$**.** Also $\varnothing \neq \{\varnothing\}$ because $\{\varnothing\}$ is a set with one element, namely the null set.

Let us now consider the subsets of the set **C = {1,2,3,4}.** In constructing a subset of **C**, we have, for each member **x** of **C,** two distinct choices: either include it in the set, or exclude it. Consequently, there are **2 x 2 x 2 x 2** choices, resulting in $2^4 = 16$ subsets of **C**. These include the empty set and the set **C** itself.

If **A** is a set from universe $\mathcal{U}$, the power set of **A**, denoted ***P*(A)** is the collection (or set) of all subsets of **A**.

Example

For the set **C** given above, ***P*(C)= {**$\varnothing$,
{1},{2},{3},{4},{1,2},{1,3},{1,4},{2,3},{2,4},{3,4},{1,2,3},{1,2,4},{1,3,4},{2,3,4},C}

In general, for any finite set **A** with $|A| = n \geq 0$, **A** has 2^n subsets and that $|P(A)| = 2^n$

Some Useful Sets

- **a)** **Z** = the set of integers **= {0,1,-1,2,-2,3,-3,…}**
- **b)** **N** = the set of non-negative integers of natural numbers = **{0,1,2,3,…}**
- **c)** Z^+ = the set of positive integers = **{1,2,3,…}**
- **d)** **Q** = the set of rational numbers = $\{a/b \mid a,b \in Z, b \neq 0\}$
- **e)** Q^+ = the set of positive rational numbers = $\{r \mid r \in Q, r>0)$
- **f)** **R** = the set of real numbers
- **g)** R^+ = the set of positive real numbers

1.5 Set Operations and the Laws of Set Theory

For $A,B \subseteq \mathcal{U}$ we define the following:

a) $A \cup B$ (the *union* of **A** and **B**) = $\{x \mid x \in A \vee x \in B\}$
b) $A \cap B$ (the *intersection* of **A** and **B**) = $\{x \mid x \in A \wedge x \in B\}$
c) $A \Delta B$ (the *symmetric difference* of A and B) = $\{x \mid (x \in A \vee x \in B) \wedge x \notin A \cap B\}$ = $\{x \mid x \in A \cup B \wedge x \notin A \cap B\}$

Note that if , $A,B \subseteq \mathcal{U}$ then $A \cup B, A \cap B, A \Delta B \subseteq \mathcal{U}$ .Consequently, $\cup$, $\cap$ and Δ are closed binary operations on $P(\mathcal{U})$ or that $P(\mathcal{U})$ is *closed* under these binary operations**.**

Examples

With $\mathcal{U}$ = **{1,2,3,4,5,6,7,8,9,10}, A = {1,2,3,4,5}, B= {3,4,5,6,7}** and **C = {7,8,9}** we have**:**

- **a)** $A \cap B = \{3,4,5\}$
- **b)** $A \cup B = \{1,2,3,4,5,6,7\}$
- **c)** $B \cap C = \{7\}$
- **d)** $A \cap C = \varnothing$
- **e)** $A \Delta B = \{1,2,6,7\}$
- **f)** $A \cup C = \{1,2,3,4,5,7,8,9\}$
- **g)** $A \Delta C = \{1,2,3,4,5,7,8,9\}$

Definition

Let **A,B** $\subseteq \mathcal{U}$. The sets **A** and **B** are called *disjoint*, or *mutually disjoint,* when $\mathbf{A} \cap \mathbf{B} = \emptyset$**.**

For a set $\mathbf{A} \subseteq \mathcal{U}$, the *complement* of **A**, denoted $\mathcal{U} - \mathbf{A}$, or $\overline{\mathbf{A}}$ is given by $\{\mathbf{x} \mid \mathbf{x} \in \mathcal{U} \wedge \mathbf{x} \notin \mathbf{A}\}$

Examples

Referring to sets **A,B,C** above:

$\overline{\mathbf{A}} = \{6,7,8,9,10\}$, $\overline{\mathbf{B}} = \{1,2,8,9,10\}$ and $\overline{\mathbf{C}} = \{1,2,3,4,5,6,10\}$

Definition

Let **A,B** $\subseteq \mathcal{U}$. the *relative complement* of **A** in **B**, denoted **B-A**, is given by $\{\mathbf{x} \mid \mathbf{x} \in \mathbf{B} \wedge \mathbf{x} \notin \mathbf{A}\}$.

Examples

Referring to sets **A,B,C** above:

a) **B-A = {6,7}**
b) **A-B = {1,2}**
c) **A-C = A**
d) **C-A = C**
e) **A-A =** $\emptyset$
f) $\mathcal{U} - \mathbf{A} = \overline{\mathbf{A}}$

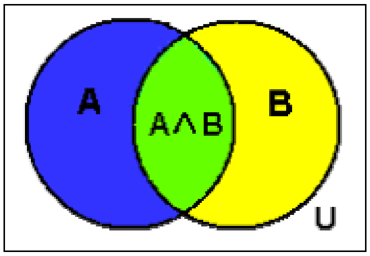
The Laws of Set Theory

For any sets **A,B,** and **C** taken from a universe $\mathcal{U}$

1. $\overline{\overline{\mathbf{A}}} = \mathbf{A}$	Law of *Double Complement*
2. $\overline{\mathbf{A} \cup \mathbf{B}} = \overline{\mathbf{A}} \cap \overline{\mathbf{B}}$ $\overline{\mathbf{A} \cap \mathbf{B}} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$	De Morgan's Laws
3. $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A}$ $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$	Commutative Laws
4. $\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}$ $\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$	Associative Laws
5. $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$ $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$	Distributive Laws
6. $\mathbf{A} \cup \mathbf{A} = \mathbf{A}$ $\mathbf{A} \cap \mathbf{A} = \mathbf{A}$	Idempotent Laws
7. $\mathbf{A} \cup \mathcal{U} = \mathcal{U}$ $\mathbf{A} \cap \mathcal{U} = \mathbf{A}$	Identity laws
8. $\mathbf{A} \cup \overline{\mathbf{A}} = \mathcal{U}$ $\mathbf{A} \cap \overline{\mathbf{A}} = \emptyset$	Inverse Laws
9. $\mathbf{A} \cup \mathcal{U} = \mathcal{U}$ $\mathbf{A} \cap \emptyset = \emptyset$	Domination laws

10. A ∪ (A ∩ B) = A Absorption Laws
A ∩ (A ∪ B) = A

When we consider the relationships that exist between sets, we can investigate this graphically using something called a *Venn* [1]*diagram.* A Venn diagram is constructed as follows: $\mathscr{U}$ is depicted as the interior of a rectangle, while subsets of $\mathscr{U}$ are represented by the interiors of circles.



The Venn diagram is a useful way of establishing set equalities. Another technique for establishing set equalities is the *membership table.*

We observe that for sets **A,B** $\subseteq \mathscr{U}$, an element **x** $\in \mathscr{U}$**,**satisfies exactly one of the following four situations:

a) $x \notin A, x \notin B$
b) $x \notin A, x \in B$
c) $x \in A, x \notin B$
d) $x \in A, x \in B$

This can then be written in tabular form, using a 0 when x is not in a set and a 1 when it is in a set.

A	**B**	**A ∩B**	**A ∪ B**
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Example

Use *Membership Tables* to establish the equality of **A ∪ (B ∩ C)** with **(A ∪ B) ∩ (A ∪ C)**

A	**B**	**C**	**(B ∩ C)**	**A ∪ (B ∩ C)**	**A ∪ B**	**A ∪ C**	**(A ∪ B) ∩ (A ∪ C)**
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

[1] Named after the English Logician John Venn (1834-1923)

Set Theory – Tutorial Exercises

1. Which of the following sets are equal?

  a) **{1,2,3}** b) **{3,2,1,3}** c) **{3,1,2,3}** d) **{1,2,2,3}**

2. Let $\mathbf{A = \{1, \{1\}, \{2\}\}}$. Which of the following statements are true?

  a) $\mathbf{1 \in A}$ b) $\mathbf{\{1\} \in A}$ c) $\mathbf{\{1\} \subseteq A}$ d) $\mathbf{\{\{1\}\} \subseteq A}$

  e) $\mathbf{\{2\} \in A}$ f) $\mathbf{\{2\} \subseteq A}$ g) $\mathbf{\{\{2\}\} \subseteq A}$ h) $\mathbf{\{\{2\}] \supset A}$

3. Repeat question 2 using the set $\mathbf{A = \{1,2,\{2\}\}}$

4. Determine all the elements in each of the following sets:

  a) $\{1 + (-1)^n \mid n \in N\}$

  b) $\{n + (1/n) \mid n \in \{1,2,3,5,7\}\}$

  c) $\{n^3 + n^2 \mid n \in \{0,1,2,3,4\}\}$

  d) $\{1/(n^2 + n) \mid$ **n is an odd positive integer and** $n \le 11\}$

5. For $\mathbf{A= \{1,2,3,4,5,6,7\}}$, determine the number of

  a) Subsets of **A**
  b) Nonempty subsets of **A**
  c) Proper subsets of **A**
  d) Nonempty proper subsets of **A**
  e) Subsets of **A** containing 3 elements
  f) Subsets of **A** containing 1,2
  g) Subsets of **A** containing 5 elements including 1,2
  h) Proper subsets of **A** containing 1,2
  i) Subsets of **A** with an even number of elements
  j) Subsets of **A** with an odd number of elements
  k) Subsets of **A** with an odd number of elements, including the element 3

6. For $\mathcal{U}$= {1,2,3,4,5,6,7,8,9,10}, let A={1,2,3,4,5}, B={1,2,4,8}, C={1,2,3,5,7} and D={2,4,6,8}. Determine each of the following:

  a) $\mathbf{(A \cup B) \cap C}$

  b) $\mathbf{A \cup (B \cap C)}$

  c) $\mathbf{\overline{C} \cup \overline{D}}$

  d) $\mathbf{\overline{C \cap D}}$

  e) $\mathbf{(A \cup B) - C}$

  f) $\mathbf{A \cup (B - C)}$

  g) $\mathbf{(B - C) - D}$

  h) $\mathbf{B - (C - D)}$

  i) $\mathbf{(A \cup B) - (C \cap D)}$

7. Let $\mathcal{U}$= **{a,b,c,…x,y,z}** with **A={a,b,c}** and **C={a,b,d,e}.** If $\mathbf{|A \cap B| = 2}$ and $\mathbf{(A \cap B) \subset B \subset C}$, determine **B**.

8. Using Venn diagrams or Membership Tables, investigate the truth or falsity of the each of the following for sets $\mathbf{A,B,C} \subseteq \mathcal{U}$.

  a) $\mathbf{A \,\Delta\, (B \cap C) = (A \,\Delta\, B) \cap (A \,\Delta\, C)}$
  b) $\mathbf{A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)}$

9. A supermarket discovers that from a sample of 50 shoppers, 30 buy tea, 25 buy coffee and 10 buy both coffee and tea. How many shoppers buy either coffee or tea.? (Hint – use Venn Diagrams)